Proceeding Series of the Brazilian Society of Applied and Computational Mathematics, Vol. 3, N. 1, 2015.
Trabalho apresentado no XXXV CNMAC, Natal-RN, 2014.

The Primal and Dual Problems of b-Complementary Multisemigroup.

Eleazar Madriz
Universidade Federal do Recôncavo da Bahia
Cruz da Almas, BA, Brazil,
egmlozada@gmail.com

May 12, 2014

Abstract

In this work we present the primal and dual problem for a b-complementary multisemigroup.

1 Introduction

In general, for the master additive system problem (see [1],[2]) there exits results of lifting facets that can't be used for non-master because when projecting a facet of a master polyhedra onto a non-master polyhedra we don't obtain a facet of the non-master. In the case where the algebraic structure is an abelian group, Gastou (see [3]) showed how to lift in a sequential way. This way does not consider the dual problem of the problem associated in order to caracterize the facets, this is a motive for the research. The main purpose of this paper is to define the dual problem of a primal problem for a b-complementary multisemigroup. We have extended the result in [5] the semigroup for the b-complementary multisemigroup.

2 The b-complementary Multisemigroup.

An *additive system* $(A, \widehat{+})$ is defined to be a non-empty finite set A together with addition $\widehat{+} : 2^A \times 2^A \to 2^A$ $(2^A = \{H : H \subset A\})$ such that:

(1) $\{g\}\widehat{+}\{h\} \subseteq A$, for all g and h in A;

(2) $S\widehat{+}T = \cup_{s\in S,t\in T}(\{s\} \widehat{+} \{t\})$, for all $S, T \subseteq A$.

In this work we use $g\widehat{+}h$ in the place of $\{g\}\widehat{+}\{h\}$.

An *identity* is an element of A denoted $\widehat{0}$ such that $\widehat{0}\widehat{+}g = g\widehat{+}\widehat{0}$, for all $g \in A$, a *infinity*, ∞, is the element of A such that $\infty\widehat{+}S = S\widehat{+}\infty = \infty$, for all $S \in 2^A$.

DOI: 10.5540/03.2015.003.01.0423

We assume that both identity and infinity, are unique if they exist.The set A_+ of *proper elements* is the set $A\backslash\{\widehat{0},\infty\}$. In this work we denote N by the positive integer set and R as the set of real numbers.

An additive system is *associative* if :

$$(S\widehat{+}T)\widehat{+}U = S\widehat{+}(T\widehat{+}U)$$

for all $S,T,U \in 2^A$. And, an additive system is *abelian* if $S\widehat{+}T = T\widehat{+}S$, for all $S,T \in 2^A$.

An *expression* of the additive system $(A,\widehat{+})$ is defined recursively as follows (see [4]):

(i) (ξ) is an *empty expression*;

(ii) (g) is *a primitive expression*, for all $g \in A$;

(iii) $E = (E_1\widehat{+}E_2)$ is an *expression*, for all expressions E_1 and E_2,

For E as (iii) we call the expressions E_1 and E_2 the *subexpression* of E.

The *evaluation* γ (see [4]) is the function of the expressions of $(A,\widehat{+})$ in the 2^A defined recursively by

- If $E = (\xi)$ then $\gamma(E) = \{\widehat{0}\}$;
- If $E = (g)$ then $\gamma(E) = \{g\}$, for all $g \in A$;
- If $E = (E_1\widehat{+}E_2)$ then $\gamma(E) = \gamma(E_1)\widehat{+}\gamma(E_2)$.

The *incident vector* of an expression E is the vector $t = (t(g); g \in A_+)$ such that $t(g)$ is the number of times that (g) appears as the primitive subexpression of E. Now, let b be a fixed element in an additive system $(A,\widehat{+})$, the expression E is a *solution expression* of b if $b \in \gamma(E)$. And $t \in \mathsf{N}^{A_+}$ is a *solution vector* of b if there are a solution expression E of b.

Let $(A,\widehat{+})$ be an abelian associativity additive system. Since $(A,\widehat{+})$ is associative and abelian, for any positive integer k and any $g \in A$, we can defined kg by

$$kg = \gamma((\overbrace{(g)\widehat{+}..\widehat{+}(g)}^{k-times})).$$

Now, since there are only a finite number of subsets of A in the sequence of sets $0g, 1g, 2g, \ldots, kg, \ldots$ there are sets which appear an infinite number of times, such sets are called *loop sets of g*. The *loop of* g is the union of all the loop sets of g. Let $s = mg$ be the first occurrence of any set appearing for the second time in the sequence $(kg \mid k \geq 0)$. Since s appears the second time in the sequence, $s = pg$ for some $p < m$, and the sequence of distinct sets $(kg \mid p < k \leq m-1)$ is the same as $(kg \mid m \leq k \leq 2m-p-1)$. In fact we have $(p+k+il)g = (p+kg)$ (where $l = m-p$) for $0 \leq k \leq l-1$ and $i \geq 0$, since $(m+k)g = mg\widehat{+}kg = pg\widehat{+}kg = (p+k)g$. The *loop order* of g is defined to be this l. Clearly $h \in A$ is in loop of g if and only if there exists $k \geq 0$ such that $h \in (k+il)g$ for all $i \geq 0$, where l is the loop order of g.

DOI: 10.5540/03.2015.003.01.0423

Let $(A, \widehat{+})$ be an abelian associative additive system and $b \in A_+$. We say that $(A, \widehat{+})$ is $b-$*consistent*, if and only if $b \in b\widehat{+}kg$, for all $g \in A$ and for all $k \in Z_+$, and that $(A, \widehat{+})$ is a *multisemigroup* if it's g-consistent for all $g \in A$.

Let $(A, \widehat{+})$ be an additive system and $b \in A$, we define

$$b \sim g = \{x \in A : b \in x\hat{+}g\}.$$

These sets induce a partial order in A, we say $g\tilde{<}h$ when $b \sim g \subseteq b \sim h$. When the set $b \sim g$ has a minimum element, this minimum element is called $b-$*complement* of g. The and is denoted by $\widehat{g}$. A additive system A is called *b-complementary* when every element has a $b-$*complement*. An element $g \in A$ is *infeasible* whenever there is not solution of the equation $b \in g\hat{+}x$, that is , $b \sim g = \emptyset$. We can assume, without loss of generality, that the additive system has at most one infeasible element denoted by $\widehat{\infty}$.

3 The Optimization Problem.

Given a finite b-complementary multisemigroup A and a subset M of A, the b-complementary multisemigroup problem defined as

$$min \sum_{g \in M} c(g)t(g)$$

$$\text{s.t: } b \in \widehat{\sum_{g \in M}} t(g)g$$

$$t \in \mathsf{N}^M$$

where $c(g) \in \mathsf{R}$ for all $g \in M$.

The function $\pi : A \to R$ has *subadditivity* (see [2]) if it satisfies:

(i) $\pi(\emptyset) = -\infty$;

(ii) $\pi(G) = \max\{\pi(g) : g \in G\}$ for all $G \subseteq A$;

(iii) $\pi(\left\{\widehat{0}\right\}) = 0$;

(iv) $\pi(G) + \pi(H) \geq \pi(G\widehat{+}H)$ for all $G, H \subseteq A$.

The *Subadditivity Cone* is the set

$$C(A) = \{(\pi(g); g \in A_+) : \pi \text{ is a subadditivity function } \}$$

We denote the linearity of $C(A)$ by $L(A)$, $\pi(\{g\})$ by $\pi(g)$ and (L, E) as a base of $C(A)$ (see [6]).

DOI: 10.5540/03.2015.003.01.0423

4 Duality

Consider the following linear programming problem

$$\min\ \widetilde{c}x \tag{1}$$
$$\text{s.t: } \widetilde{A}x = \widetilde{b} \tag{2}$$
$$\widetilde{E}x \geq \widetilde{h} \tag{3}$$
$$x \geq 0 \tag{4}$$

where x and c are an n vetor, $\widetilde{b}$ is an m vetor, $\widetilde{h}$ is an p vetor, A is an m by n matrix and E is a p by n matrix. Corresponding to this problem, called *primal problem*, consider the following linear problem

$$\max\ \widetilde{\pi}\widetilde{b} + \widetilde{\mu}\widetilde{h} \tag{5}$$
$$\text{s.t: } \widetilde{\pi}\widetilde{A} + \widetilde{\mu}\widetilde{E} \leq \widetilde{c} \tag{6}$$
$$\widetilde{\pi} \text{ unrestricted, and } \widetilde{\mu} \geq 0 \tag{7}$$

where $\widetilde{\pi}$ and $\widetilde{\mu}$ are row vetor of size m and p, respectively. This problem is called the *dual problem* of the primal problem (1)-(4) (see 2.5 in [7]).

Now, for $(A, \widehat{+})$ be a b-complementary multisemigroup, let $P(A, b)$ the convex hull of the set $\{t \in \mathsf{N}^{A_+} : b \in \widehat{\sum}_{g\in A_+} t(g)g\}$. We denote by P the following linear programming problem

$$min \sum_{g\in A_+} c(g)t(g)$$

$$\text{s.t: } t \in P(A, b)$$

where $c(g) \in \mathsf{R}$ for all $g \in A_+$.

In ([6]) Araoz and Johnson show the following theorem:

Theorem 4.1 *[2, Theorem 3.8] Let (L, E) be a base of $C(A)$. The following system defined a $P(A, b)$*

$$\sum_{g\in A_+} \rho(g)t(g) = \rho(b),\ \textit{for all } \rho \in L \tag{8}$$
$$\sum_{g\in A_+} \pi(g)t(g) \geq \pi(b),\ \textit{for all } \pi \in E \tag{9}$$
$$t(g) \geq 0,\ \textit{for all } g \in A_+. \tag{10}$$

In order to defined the dual problem of the problem P we shown the following theorem.

DOI: 10.5540/03.2015.003.01.0423

Theorem 4.2 *The P problem is equivalent to the P_p problem*

$$\min \sum_{g\in A_+} c(g)t(g) \tag{11}$$

$$\sum_{g\in A_+} \rho(g)t(g) = \rho(b), \quad \rho \in L; \tag{12}$$

$$\sum_{g\in A_+} \pi(g)t(g) \geq \pi(b), \quad \pi \in E; \tag{13}$$

$$t(g) \geq 0, \quad g \in A_+, \tag{14}$$

where (L, E) is a base for $C(A)$ and $c \in R^{A_+}$

Proof. By theorem 4.1 the system (8)-(10) defined a $P(A, b)$, then the problem P and P_p are equivalent.$\diamondsuit$

Theorem 4.3 *The dual problem of P_p is the problem P_d*

$$max \sum_{\rho\in L} \rho(b)v(\rho) + \sum_{\pi\in E} \pi(b)w(\pi) \tag{15}$$

$$\sum_{\rho\in L} \rho(g)v(\rho) + \sum_{\pi\in E} \pi(g)w(\pi) \leq \quad c(g), \quad g \in A_+ \tag{16}$$

$$v(\rho) \ \textit{unrestricted}, \ \rho \in L \tag{17}$$

$$w(\pi) \geq 0, \pi \in E. \tag{18}$$

Proof. Since L and E are finite sets (see [6]), the system (12)-(14) is the form of the system (2)-(4) where $n = |A_+|$, $m = |L|$, $p = |E|$, $(\widetilde{A})_{\rho,g} = \rho(g)$ for $\rho \in L, g \in A_+$, $(\widetilde{E})_{\pi,g} = \pi(g)$ for $\rho \in E, g \in A_+$, $\widetilde{b} = (\rho(b) : \rho \in L)$,$\widetilde{h} = (\pi(b) : \pi \in E)$ and $\widetilde{c} = (c(g) : g \in A_+)$. Therefore the dual problem of P_p is the form by duality linear programming the proof is complete $\diamondsuit$

References

[1] Aráoz J. and Johnson E., ”Polyhedra of Multivalued System Problems”, Report No.82229-OR, Institut für Ökonometrie und Operations Research, Bonn, W. Germany (1982)

[2] Aráoz J. and Johnson E.,”Morphic Liftings between of pairs of Integer Polyhedra”,Research Report 89616OR, Int. Operations Research, Bonn 1989.

[3] Gastou G., ”On facets of integer Programming Polyhedra”, Ph.D. Thesis, Administrative Science Dept, Yale University, New Haven, 1982.

[4] Gomory R., ”Some Polyhedra Related to Combinatorial Problems”, Journal of Linear Algebra and Its Applications, 2(4),451-558 (1969).

DOI: 10.5540/03.2015.003.01.0423

[5] Johnson E.L.,"Facets, Subadditivity, and Duality for Group and Semi-group Problems", Society for Industrial and Applied Mathematics 19103, Philadelphia,USA 1980.

[6] Rockafellar R. T., "Convex Analysis", Princenton Univerty Press, Princenton, New Jersey, U.S.A (1969)

[7] Salkin H. and Mat., "Foundations of Integer Programming", Elsevier Science Publishing, North-Holland (1989)

DOI: 10.5540/03.2015.003.01.0423