

An Approach to Interval Fuzzy Probabilities

Tiago C. Asmus, **Benjamin Bedregal ,**

PPgSC, DIMAP, Universidade Federal do Rio Grande do Norte
Campus Universitário, 59078-970, Natal, Brazil
E-mail: tiagoasmus@gmail.com, brcbedregal@gmail.com

Graçaliz P. Dimuro

PPDCOMP, C3, Universidade Federal do Rio Grande
Av. Itália km 08, Campus Carreiros, 96203-090, Rio Grande, Brazil
E-mail: gracaliz@gmail.com

Abstract: *Fuzzy sets and logic have been largely applied used to the treatment of the uncertainty, vagueness and ambiguity found in the modeling of real problems. However, there may exist also the case when there is uncertainty related to the membership functions to be used in the modeling of fuzzy sets and fuzzy numbers, as there are many ways to define the shape of this kind of number. Thus, one can use, for example, the theory of interval fuzzy sets to address this uncertainty, considering different modeling of fuzzy numbers into a single interval fuzzy number. In this paper, we use interval fuzzy numbers to represent probabilities that are difficult to be estimated and where the modeling of fuzzy numbers is not trivial. To elaborate the calculation of probabilities, it will be introduced an approach based on one used by Buckley and Eslami, where the probabilities respect an arithmetic restriction. We discuss several properties of the proposed approach.*

Palavras-chave: *Interval Fuzzy Logic, Interval Fuzzy Sets, Interval Fuzzy Probabilities*

1 Introduction

In problems of decision making in an environment with uncertainty, the decision maker must estimate the probabilities of different actions, which may lead to different outcomes. However, sometimes these probabilities are difficult to estimate precisely, such as problems in agent-based social simulation [1, 12, 14], which often have linguistic variables to define some parameters of the agents involved in the modeling [2, 11, 15] under vagueness, ambiguity and uncertainty.

However, one can find in the theory of fuzzy sets an alternative to model this imprecision, as fuzzy numbers are ideal for representing linguistic variables and modeling imprecise values [16, 17]. But still, there may be uncertainty in how to model this fuzzy numbers, as there are many ways to define the shape of this kind of number. Thus, one can use the theory of interval fuzzy sets to address this uncertainty, considering different modeling of fuzzy numbers into a single interval fuzzy number [9].

In this paper, we use interval fuzzy numbers to represent probabilities that are difficult to be estimated and where the modeling of fuzzy numbers is not trivial. To elaborate the calculation of probabilities, it will be introduced an approach based on one used by Buckley and Eslami [6], where the probabilities respect an arithmetic restriction on the interval $[0, 1]$. We discuss several properties of the proposed approach.

The paper is organized as follows. In Section 2, preliminary aspects and the aforementioned approach for fuzzy probabilities is presented, as the definition of fuzzy mean¹. In Section 3, the focus is on development of the an approach to interval fuzzy probabilities, joint with some propositions and proper-

¹For the lack of space, we omit some proofs of propositions and properties, which can be found in [3].

ties, and the definition of interval fuzzy mean. Section 4 brings the final considerations of this paper and some ideas for future work.

2 Fuzzy Probabilities

For the purpose of this paper, we consider triangular and symmetrical fuzzy numbers, denoted by $\bar{F} = (a/u/b)$, with u being the core of the fuzzy number, and its α -cuts represented by $\bar{F}[\alpha] = [(u - a)\alpha + a, (u - b)\alpha + b]$. Interval fuzzy numbers [10] can be represented by pairs of generator fuzzy numbers as $\hat{F} = (\bar{F}_i, \bar{F}_s)$, where \bar{F}_i and \bar{F}_s are defined by the lower and upper membership functions of \hat{F} , respectively. The $[\alpha_1, \alpha_2]$ -cuts of \hat{F} are given by $\hat{F}[\alpha_1, \alpha_2] = \bar{F}_i[\alpha_1] \cap \bar{F}_s[\alpha_2]$. For ordering interval fuzzy numbers, we adopt the Interval AD-Order [4, 5].

We present a general definition, through stochastic vectors, for the approach proposed by Buckley to calculate fuzzy probabilities [6, 7], which does not consider the standard probability theory [18]. Let $X = \{x_1, \dots, x_n\}$ be a finite set and $P : \wp(X) \rightarrow [0, 1]$ a probability function defined for all subsets of X , with $P(\{x_i\}) = \phi_i$, $1 \leq i \leq n$, $0 \leq \phi_i \leq 1$ and $\sum_{i=1}^n \phi_i = 1$. Thus, (X, P) characterizes a finite and discrete probability distribution, represented by the set $\Phi = \{\phi_1, \dots, \phi_n\}$.

We observe that the elements of Φ are often obtained by experts' opinions, not always representing precise values or consensus. To model this uncertainty, one may use fuzzy numbers $\bar{\phi}_i$, forming a new set denoted by $\bar{\Phi}$. Thus, the fuzzy probability function \bar{P} is the representation of an uncertain expert opinion on the studied subject. The elements of \bar{P} are symmetric triangular fuzzy numbers $\bar{P}(\{x_i\}) = \bar{\phi}_i = (a_i/u_i/b_i)$, for $1 \leq i \leq n$, where a_i and b_i are the pessimistic and optimistic estimates of x_i , respectively, and u_i is the "the most likely" estimate.

When several experts provide these values, then a_i and b_i are obtained by calculating the variance of all the pessimistic and optimistic estimates, respectively, on the occurrence of x_i , and u_i is the result of the arithmetic mean of all "most likely" estimates about the analyzed event [6, 7]. Hardly the obtained triangular fuzzy number is going to be symmetric, then one should transform this fuzzy number in order to obtain a new symmetric triangular fuzzy number denoted by $(a_i * /u_i * / * b_i)$, where $u_i^* - a_i^* = \min(u_i - a_i, b_i - u_i)$, with $u_i^* - a_i^* = b_i^* - u_i^*$.

For $(X, \bar{\Phi})$ to represent a finite and discrete distribution of probabilities, it is necessary to make an arithmetic restriction, since the interval sum of the α -cuts of all $\bar{\phi}_i$ hardly results in 1 [6]. Thus, for all $0 \leq \alpha \leq 1$, we choose a e_i from each α -cut $\bar{\phi}_i[\alpha]$ so that $\sum_{i=1}^n e_i = 1$. So, instead of the interval sum of the α -cuts necessarily results in 1, we just need to obtain a single crisp value e_i from each $\bar{\phi}_i[\alpha]$ in a way that the sum of all chosen e_i results in 1.

For that, consider the set of stochastic vectors $E = \{(e_1, \dots, e_n) \in [0, 1]^n \mid \sum_{i=1}^n e_i = 1\}$, the set $\mathcal{A}_{\bar{\Phi}}^\alpha = \bar{\phi}_1[\alpha] \times \dots \times \bar{\phi}_n[\alpha]$, which is the cartesian product of the elements of $\bar{\Phi}$, represented by α -cuts. The set $\mathcal{S}_{\bar{\Phi}}^\alpha = \mathcal{A}_{\bar{\Phi}}^\alpha \cap E$ is the domain of the functions that characterize the intended arithmetic restriction. Consider a subset $A \subseteq X$ and let $I_A^X = \{i \in \{1, \dots, n\} \mid x_i \in A\}$ be the set of indexes of the elements of A related to the indexes of the set $X = \{x_1, \dots, x_n\}$.² For $0 \leq \alpha \leq 1$, the function $f_{\bar{\Phi}, \alpha}^A : \mathcal{S}_{\bar{\Phi}}^\alpha \rightarrow [0, 1]$, given by $f_{\bar{\Phi}, \alpha}^A(e_1, \dots, e_n) \mapsto \sum_{i \in I_A} e_i$, defines the desired arithmetic restriction. Then, the fuzzy probability of a subset $A \subseteq X$ is obtained through its α -cuts:

$$\bar{P}(A)[\alpha] = \left\{ f_{\bar{\Phi}, \alpha}^A(e) \mid e = (e_1, \dots, e_n) \in \mathcal{S}_{\bar{\Phi}}^\alpha \right\}. \tag{1}$$

Lemma 1. *The set $\mathcal{S}_{\bar{\Phi}}^\alpha$ is formed by connected, closed and bounded intervals.*

Proposition 1. *$\bar{P}(A)$ is a symmetric triangular fuzzy number.*

Considering A and B as subsets of X , $\bar{0} = (0/0/0)$ and $\bar{1} = (1/1/1)$, and using the AD-order [5, 4], we have the following properties:

a) $A \cap B = \emptyset \Rightarrow \bar{P}(A) + \bar{P}(B) \leq \bar{P}(A \cup B)$.

²When the set of indexes of the elements of X can be understood by context, then we denote the set of indexes from $A \subseteq X$ simply as I_A .

- b) $A \subseteq B \Rightarrow \bar{P}(A) \leq \bar{P}(B)$.
- c) $\forall A : \bar{0} \leq \bar{P}(A) \leq \bar{1}$;
- d) $\bar{P}(\emptyset) = \bar{0} \wedge \bar{P}(X) = \bar{1}$;
- e) $\bar{P}(A) + \bar{P}(A') \leq \bar{1}$;
- f) $A \cap B \neq \emptyset \Rightarrow \bar{P}(A) + \bar{P}(B) - \bar{P}(A \cap B) \leq \bar{P}(A \cup B)$.

Finally, we show how to define the fuzzy mean $\bar{\mu}_{\bar{\Phi}}^r$ through the multiplication of each $\bar{\Phi} = \{\bar{\phi}_1, \dots, \bar{\phi}_n\}$ by the real values in $r = (r_1, \dots, r_n) \in \mathbb{R}^n$. To this end, we introduce the function $g_{\bar{\Phi}, \alpha}^r : \mathcal{S}_{\bar{\Phi}}^\alpha \rightarrow \mathbb{R}$, defined by $g_{\bar{\Phi}, \alpha}^r(e_1, \dots, e_n) = \sum_{i=1}^n e_i \cdot r_i$, for $0 \leq \alpha \leq 1$, which ensures that the arithmetic restriction is respected. The fuzzy mean for a $r = (r_1, \dots, r_n) \in \mathbb{R}^n$ is defined through its α -cuts, for $0 \leq \alpha \leq 1$:

$$\bar{\mu}_{\bar{\Phi}}^r[\alpha] = \left\{ g_{\bar{\Phi}, \alpha}^r(e) \mid e = (e_1, \dots, e_n) \in \mathcal{S}_{\bar{\Phi}}^\alpha \right\}. \tag{2}$$

Proposition 2. $\bar{\mu}_{\bar{\Phi}}^r$ is a triangular symmetric fuzzy number.

In the literature, we can find other approaches to fuzzy probabilities. For example, the approach proposed by Costa [8], without the use of an arithmetic restriction, facilitates the calculation of each probability, avoiding the interdependence between them, but it produces fuzzy numbers with greater dispersion than the ones produced by our approach. As the probabilities defined in this paper are calculated to be utilized in problems of decision making, we prefer to work with fuzzy values with the least possible dispersion to avoid ambiguous cases in the ordering of imprecise values.

3 Interval Fuzzy Probabilities

We adopt an analogous approach for fuzzy probabilities in order to define interval fuzzy probabilities. Let $X = \{x_1, \dots, x_n\}$ be a finite set and $P : \wp(X) \rightarrow [0, 1]$ a probability function defined for all subsets of X , for $P(\{x_j\}) = \phi_j$, $1 \leq j \leq n$, $0 \leq \phi_j \leq 1$ and $\sum_{i=1}^n \phi_j = 1$, characterizing a discrete and finite probability distribution $\Phi = \{\phi_1, \dots, \phi_n\}$.

Consider that some of these ϕ_j represent imprecise values, in a way that even its fuzzy modeling is not trivial. Thus, we use interval fuzzy numbers substituting each ϕ_j for $\hat{\phi}_j$, constituting a new set $\hat{\Phi}$. Then, we have a interval fuzzy probability function \hat{P} , in a way that the elements of $\hat{\Phi}$ are represented by symmetric triangular interval fuzzy number $\hat{P}(x_1) = \hat{\phi}_j = (\bar{\phi}_{j_i}, \bar{\phi}_{j_s})$, for $1 \leq j \leq n$, where $\bar{\phi}_{j_i}$ represents the fuzzy modeling with the lowest imprecision, and $\bar{\phi}_{j_s}$ represents the fuzzy modeling with the highest imprecision, according to the experts' opinions.

For $(X, \hat{\Phi})$ to represent a finite and discrete probability function, we define an arithmetic restriction. For $0 \leq \alpha_1 \leq \alpha_2 \leq 1$, we choose, from each $[\alpha_1, \alpha_2]$ -cut $\hat{\phi}_j[\alpha_1, \alpha_2]$, an e_j , so that $\sum_{j=1}^n e_j = 1$. Consider the set of estocastic vectors $E = \left\{ (e_j, \dots, e_n) \in [0, 1]^n \mid \sum_{j=1}^n e_j = 1 \right\}$, the set $\mathcal{A}_{\hat{\Phi}}^{\alpha_1, \alpha_2} = \hat{\phi}_1[\alpha_1, \alpha_2] \times \dots \times \hat{\phi}_n[\alpha_1, \alpha_2]$ of the cartesian product of the $[\alpha_1, \alpha_2]$ -cuts of the elements of $\hat{\Phi}$, and $\mathcal{S}_{\hat{\Phi}}^{\alpha_1, \alpha_2} = \mathcal{A}_{\hat{\Phi}}^{\alpha_1, \alpha_2} \cap E$, which is the domain of the function that characterizes the arithmetic restriction. Let $A \subseteq X$ be the index set of the elements of A , related to the indexes from X , defined by $J_A^X = \{j \in \{1, \dots, n\} \mid x_j \in A\}$.³ The function $f_{\hat{\Phi}, [\alpha_1, \alpha_2]}^A : \mathcal{S}_{\hat{\Phi}}^{\alpha_1, \alpha_2} \rightarrow [0, 1]$, which characterizes the arithmetic restriction, is defined by:

$$f_{\hat{\Phi}, [\alpha_1, \alpha_2]}^A(e_1, \dots, e_n) = \sum_{j \in J_A} e_j, \tag{3}$$

where $0 \leq \alpha_1 \leq \alpha_2 \leq 1$. The fuzzy probability of a subset $A \subseteq X$ is defined through its $[\alpha_1, \alpha_2]$ -cuts:

$$\hat{P}(A)[\alpha_1, \alpha_2] = \left\{ f_{\hat{\Phi}, [\alpha_1, \alpha_2]}^A(e) \mid e = (e_1, \dots, e_n) \in \mathcal{S}_{\hat{\Phi}}^{\alpha_1, \alpha_2} \right\}. \tag{4}$$

Lemma 2. The set $\mathcal{S}_{\hat{\Phi}}^{\alpha_1, \alpha_2}$ is formed by connected, closed and bounded intervals.

³When the set of indexes from X can be understood by context, we denote the subset $A \subseteq X$ simply as J_A .

Proof. The proof is immediate since the intersection of connected, closed and bounded intervals results in connected, closed and bounded interval. \square

Proposition 3. $\hat{P}(A)$ is a symmetric triangular interval fuzzy number.

Proof. It is immediate that the function $f_{\hat{\Phi},\alpha}^A$ is continuous and its domain $\mathcal{S}_{\hat{\Phi}}^{\alpha_1,\alpha_2}$, by Lemma 2, is formed by connected, closed and bounded intervals, which implies that the image of $f_{\hat{\Phi},[\alpha_1,\alpha_2]}^A$, given by $f_{\hat{\Phi},[\alpha_1,\alpha_2]}^A(\mathcal{S}_{\hat{\Phi}}^{\alpha_1,\alpha_2}) = \hat{P}(A)[\alpha_1,\alpha_2]$, is a closed and bounded interval of real numbers. Thus, $\hat{P}(A)[\alpha_1,\alpha_2]$ is the $[\alpha_1,\alpha_2]$ -cuts of a interval fuzzy set $\hat{P}(A)$. Furthermore, $f_{\hat{\Phi},[1,1]}^A(\mathcal{S}_{\hat{\Phi}}^{1,1}) = \hat{P}(A)[1,1] \neq \emptyset$, as $(\phi_1, \dots, \phi_n) \in \mathcal{S}_{\hat{\Phi}}^{1,1}$, so $\sum_{j \in J_A} \phi_j \in \hat{P}(A)[1,1]$, meaning that $\hat{P}(A)$ is indeed an interval fuzzy number. As all $\hat{\phi}_i \in \hat{\Phi}$ are symmetric triangular interval fuzzy numbers, also is $\hat{P}(A)$ [10]. \square

Proposition 4. The interval fuzzy probability $\hat{P}(A)$ may be obtained through the calculation of its fuzzy generator probabilities $\bar{P}(A)_i$ and $\bar{P}(A)_s$, as:

$$\hat{P}(A)[\alpha_1,\alpha_2] = \bar{P}(A)_i[\alpha_1] \cap \bar{P}(A)_s[\alpha_2], \tag{5}$$

for $0 \leq \alpha_1 \leq \alpha_2 \leq 1$.

Proof. The proof is immediate, since $\hat{P}(A) = (\bar{P}(A)_i, \bar{P}(A)_s)$, for $0 \leq \alpha_1 \leq \alpha_2 \leq 1$. \square

Observe that, for each j , one can denote $\hat{\phi}_j = (\bar{\phi}_{j_i}, \bar{\phi}_{j_s})$. Then, we can calculate the fuzzy probability $\bar{P}(A)_i[\alpha_1]$ through all $\bar{\phi}_{j_i}$, and the fuzzy probability $\bar{P}(A)_s[\alpha_2]$ through all $\bar{\phi}_{j_s}$. Once these fuzzy probabilities have been calculated, we only need to do the intersection of $\bar{P}(A)_i[\alpha_1]$ and $\bar{P}(A)_s[\alpha_2]$ to obtain $\hat{P}(A)[\alpha_1,\alpha_2]$, as given in the Eq. (5). So, we obtain the representation of interval fuzzy probabilities through their fuzzy generator probabilities as follows:

$$\hat{P}(A) = (\bar{P}(A)_i, \bar{P}(A)_s). \tag{6}$$

The advantage in utilizing this kind of representation is that all the properties of fuzzy probabilities presented in the Sect. 2 are valid for interval fuzzy probabilities, which are calculated through the fuzzy generator probabilities and the order relation for interval fuzzy numbers based on the AD-order. Consider $\hat{0} = (\bar{0}, \bar{0})$, $\hat{1} = (\bar{1}, \bar{1})$ and $A, B \subseteq X$. We have the following properties:

a) $A \cap B = \emptyset \Rightarrow \hat{P}(A) + \hat{P}(B) \hat{\leq} \hat{P}(A \cup B)$.

Proof. We will show that

$$\hat{P}(A)[\alpha_1,\alpha_2] + \hat{P}(B)[\alpha_1,\alpha_2] \supseteq \hat{P}(A \cup B)[\alpha_1,\alpha_2], \tag{7}$$

which means that $\hat{P}(A) + \hat{P}(B) \hat{\leq} \hat{P}(A \cup B)$, as stated by the interval AD-order. The Eq. (7) may be rewritten as follows:

$$\{\bar{P}(A)_i[\alpha_1] \cap \bar{P}(A)_s[\alpha_2]\} + \{\bar{P}(B)_i[\alpha_1] \cap \bar{P}(B)_s[\alpha_2]\} \supseteq \{\bar{P}(A \cup B)_i[\alpha_1] \cap \bar{P}(A \cup B)_s[\alpha_2]\}. \tag{8}$$

Assuming that

$$\bar{P}(A)_i[\alpha_1] + \bar{P}(B)_i[\alpha_1] \supseteq \bar{P}(A \cup B)_i[\alpha_1], \tag{9}$$

$$\bar{P}(A)_s[\alpha_2] + \bar{P}(B)_s[\alpha_2] \supseteq \bar{P}(A \cup B)_s[\alpha_2], \tag{10}$$

it follows that so the Eq. (8) is verified. The equations (9) and (10) are both true, as we are dealing with the same case discussed in the Sect. 2 for fuzzy probabilities of two disjoint sets. \square

b) $A \subseteq B \Rightarrow \hat{P}(A) \hat{\leq} \hat{P}(B)$.

Proof. One may observe that $A \subseteq B \Rightarrow I_A \subseteq I_B$. So, considering $\hat{P}(A)[\alpha_1,\alpha_2] = [a_1(\alpha), a_2(\alpha)]$ and $\hat{P}(B)[\alpha_1,\alpha_2] = [b_1(\alpha), b_2(\alpha)]$, we always have $a_1(\alpha) \leq b_1(\alpha) \wedge a_2(\alpha) \leq b_2(\alpha)$, with $0 \leq \alpha \leq 1$, meaning that $\hat{P}(A) \hat{\leq} \hat{P}(B)$, as stated by the interval AD-order. \square

- c) $\forall A : \hat{0} \leq \hat{P}(A) \leq \hat{1}$;
- d) $\hat{P}(\emptyset) = \hat{0} \wedge \hat{P}(X) = \hat{1}$.
- e) $\hat{P}(A) + \hat{P}(A') \leq \hat{1}$.
- f) $A \cap B \neq \emptyset \Rightarrow \hat{P}(A) + \hat{P}(B) - \hat{P}(A \cap B) \leq \hat{P}(A \cup B)$.

Proof. We that

$$\hat{P}(A)[\alpha_1, \alpha_2] + \hat{P}(B)[\alpha_1, \alpha_2] - \hat{P}(A \cap B)[\alpha_1, \alpha_2] \supseteq \hat{P}(A \cup B)[\alpha_1, \alpha_2], \quad (11)$$

for $0 \leq \alpha \leq 1$, which means by the interval AD-order that $\hat{P}(A) + \hat{P}(B) - \hat{P}(A \cap B) \leq \hat{P}(A \cup B)$. The Eq. (11) may be rewritten as:

$$\{\bar{P}(A)_i[\alpha_1] \cap \bar{P}(A)_s[\alpha_2]\} + \{\bar{P}(B)_1[\alpha_1] \cap \bar{P}_s[\alpha_2]\} - \{\bar{P}(A \cap B)_i[\alpha_1] \cap \bar{P}(A \cap B)_s[\alpha_2]\} \supseteq \{\bar{P}(A \cup B)_i[\alpha_1] \cap \bar{P}(A \cup B)_s[\alpha_2]\}.$$

Assuming that

$$\bar{P}(A)_i[\alpha_1] + \bar{P}(B)_i[\alpha_1] - \bar{P}(A \cap B)_i[\alpha_1] \supseteq \bar{P}(A \cup B)_i[\alpha_1], \quad (12)$$

$$\bar{P}(A)_s[\alpha_2] + \bar{P}(B)_s[\alpha_2] - \bar{P}(A \cap B)_s[\alpha_2] \supseteq \bar{P}(A \cup B)_s[\alpha_2], \quad (13)$$

we have that Eq. (11) is true. The equations (12) and (13) are also true, as they belong to the same case discussed in the Section 2 for fuzzy probabilities of two non-disjoint sets. \square

Example 1. We illustrate the property (a) through an example, showing that we may not obtain equality in that case. Let $A = \{x_1, x_2\}$, $B = \{x_3, x_4\}$ and $C = \{x_5\}$ be events (subsets) of the sample space $X = \{x_1, x_2, x_3, x_4, x_5\}$, so that $\sum_{i=1}^5 P(x_i) = 1$. All probabilities $P(x_i)$ are uncertain, except for $P(x_5) = 0.2$. One way of represent this uncertainty is through fuzzy probabilities, so let $\bar{P}(x_1) = \bar{P}(x_2) = \bar{P}(x_3) = \bar{P}(x_4) = (0.19/0.2/0.21)$ and $P(x_5) = \bar{P}(x_5) = (0.2/0.2/0.2)$. Another way of modeling these fuzzy probabilities is admit further uncertainty, letting $\bar{P}(x_1) = \bar{P}(x_2) = \bar{P}(x_3) = \bar{P}(x_4) = (0.18/0.2/0.22)$ and $\bar{P}(x_5) = (0.2/0.2/0.2)$. If there is an impasse about which modeling is preferred, one may use interval fuzzy probabilities that cover all fuzzy modelings between the chosen fuzzy generator probabilities. Then, consider the following interval fuzzy modeling:

$$\hat{P}(x_1) = \hat{P}(x_2) = \hat{P}(x_3) = \hat{P}(x_4) = ((0.19/0.2/0.21), (0.18/0.2/0.22)),$$

and

$$\hat{P}(x_5) = ((0.2/0.2/0.2), (0.2/0.2/0.2)).$$

With these interval fuzzy probabilities, $\hat{P}(A) = \hat{P}(B) = ((0.38/0.4/0.42), (0.36/0.4/0.44))$. As $\hat{P}(C) = \hat{P}(x_5) = ((0.2/0.2/0.2), (0.2/0.2/0.2)) = 0.2$, and $X = A \cup B \cup C$, the arithmetic restriction determines that $\hat{P}(A \cup B) = ((0.8/0.8/0.8), (0.8/0.8/0.8)) = 0.8$. The sum of $\hat{P}(A)$ and $\hat{P}(B)$ has no such restriction, and its obtained through the interval sum of the $[\alpha_1, \alpha_2]$ -cuts of these interval fuzzy numbers. Thus, $\hat{P}(A) + \hat{P}(B) = ((0.76/0.8/0.84), (0.72/0.8/0.88))$. Through the interval AD-order we have that $((0.76/0.8/0.84), (0.72/0.8/0.88)) \leq ((0.8/0.8/0.8), (0.8/0.8/0.8))$, and so we conclude that $A \cap B = \emptyset$ and $\hat{P}(A) + \hat{P}(B) \leq \hat{P}(A \cup B)$.

Now we introduce the calculation of the interval fuzzy mean $\hat{\mu}_{\hat{\Phi}}^r$, weighting the interval fuzzy probabilities in $\hat{\Phi} = \{\hat{\phi}_1, \dots, \hat{\phi}_n\}$ by real values in $r = (r_1, \dots, r_n) \in \mathbb{R}^n$. Similar to the calculation of the fuzzy mean, we define an auxiliary function $q_{\hat{\Phi}, [\alpha_1, \alpha_2]}^r : \mathcal{S}_{\hat{\Phi}}^{\alpha_1, \alpha_2} \rightarrow \mathbb{R}$ as:

$$q_{\hat{\Phi}, [\alpha_1, \alpha_2]}^r(e_1, \dots, e_n) = \sum_{j=1}^n e_j \cdot r_j, \quad (14)$$

with $0 \leq \alpha_1 \leq \alpha_2 \leq 1$. Thus, the interval fuzzy mean for a $r = (r_1, \dots, r_n) \in \mathbb{R}^n$ is defined through its $[\alpha_1, \alpha_2]$ -cuts as:

$$\hat{\mu}_{\hat{\Phi}}^r[\alpha_1, \alpha_2] = \left\{ q_{\hat{\Phi},[\alpha_1, \alpha_2]}^r(e) \mid e = (e_1, \dots, e_n) \in \mathcal{S}_{\hat{\Phi}}^{\alpha_1, \alpha_2} \right\} \quad (15)$$

for $0 \leq \alpha_1 \leq \alpha_2 \leq 1$ and $q_{\hat{\Phi},[\alpha_1, \alpha_2]}^r : \mathcal{S}_{\hat{\Phi}}^{\alpha_1, \alpha_2} \rightarrow \mathbb{R}$ is defined by the Eq. (14).

Proposition 5. $\hat{\mu}_{\hat{\Phi}}^r$ is a symmetric triangular interval fuzzy number.

Proof. It is immediate that the function $q_{\hat{\Phi},[\alpha_1, \alpha_2]}^r$, defined in Eq. (14) for some $r = (r_1, \dots, r_n) \in \mathbb{R}^n$, is continuous and its domain $\mathcal{S}_{\hat{\Phi}}^{\alpha_1, \alpha_2}$, as stated by the Lemma 2, is formed by connected, closed and bounded intervals, which implies that the image of $q_{\hat{\Phi},[\alpha_1, \alpha_2]}^r$, given by $q_{\hat{\Phi},[\alpha_1, \alpha_2]}^r(\mathcal{S}_{\hat{\Phi}}^{\alpha_1, \alpha_2}) = \hat{\mu}_{\hat{\Phi}}^r[\alpha_1, \alpha_2]$, is a closed and bounded interval of real numbers. Thus, $\hat{\mu}_{\hat{\Phi}}^r[\alpha_1, \alpha_2]$ represents the $[\alpha_1, \alpha_2]$ -cuts of a interval fuzzy set $\hat{\mu}_{\hat{\Phi}}^r$. Furthermore, $q_{\hat{\Phi},[1,1]}^r(\mathcal{S}_{\hat{\Phi}}^{1,1}) = \hat{\mu}_{\hat{\Phi}}^r[1, 1] \neq \emptyset$, as $(\phi_1, \dots, \phi_n) \in \mathcal{S}_{\hat{\Phi}}^{1,1}$, so $\phi_1 \cdot r_1 + \dots + \phi_n \cdot r_n \in \hat{\mu}_{\hat{\Phi}}^r[1, 1]$, and then $\hat{\mu}_{\hat{\Phi}}^r$ is an interval fuzzy number. As all $\hat{\phi}_i \in \hat{\Phi}$ are triangular symmetric interval fuzzy numbers, $\hat{\mu}_{\hat{\Phi}}^r$ is, since we only do sums and scalar products [10]. \square

Proposition 6. The interval fuzzy mean $\hat{\mu}_{\hat{\Phi}}^r$ may be obtained through the fuzzy generator means $\bar{\mu}_{\hat{\Phi}_i}^r$ and $\bar{\mu}_{\hat{\Phi}_s}^r$, for $0 \leq \alpha_1 \leq \alpha_2 \leq 1$, as:

$$\hat{\mu}_{\hat{\Phi}}^r[\alpha_1, \alpha_2] = \bar{\mu}_{\hat{\Phi}_i}^r[\alpha_1] \cap \bar{\mu}_{\hat{\Phi}_s}^r[\alpha_2]. \quad (16)$$

Proof. Similar to Proposition 4, Eq. (16) may be obtained considering $\hat{\mu}_{\hat{\Phi}}^r = (\bar{\mu}_{\hat{\Phi}_i}^r, \bar{\mu}_{\hat{\Phi}_s}^r)$ and $0 \leq \alpha_1 \leq \alpha_2 \leq 1$. \square

As $\hat{\phi}_j = (\bar{\phi}_{j_i}, \bar{\phi}_{j_s})$, we weigh the fuzzy probabilities in $\{\bar{\phi}_{1_i}, \dots, \bar{\phi}_{n_i}\}$ by real values in $r = (r_1, \dots, r_n) \in \mathbb{R}^n$, and weight $\{\bar{\phi}_{1_s}, \dots, \bar{\phi}_{n_s}\}$ by the same real values in $r = (r_1, \dots, r_n) \in \mathbb{R}^n$, through the functions $g_{\hat{\Phi}_i, \alpha_1}^r$ and $g_{\hat{\Phi}_s, \alpha_2}^r$, defined by the Eq. (14).

So, we obtain the fuzzy generator means $\bar{\mu}_{\hat{\Phi}_i}^r[\alpha_1]$ and $\bar{\mu}_{\hat{\Phi}_s}^r[\alpha_2]$ (represented by their α -cuts). By the intersection of these intervals, we obtain the interval fuzzy mean, represented by its $[\alpha_1, \alpha_2]$ -cuts.

4 Conclusion

The presented approach for interval fuzzy probabilities was shown as a viable way to represent imprecise probabilities with uncertainty in how the imprecision should be modeled. Coupled with the interval AD-order, this approach allows that certain properties of classical probability are maintained. This enables the expansion of problems where crisp probabilities are applied to versions with fuzzy or interval fuzzy probabilities.

It was observed that the interval fuzzy probabilities and interval fuzzy mean may be calculated using fuzzy generator probabilities, a simple alternative which takes advantage of the all the developed concepts for fuzzy probabilities.

Finally, for future work, we intend to use the presented approach to deal with the concept of interval fuzzy random variable and apply interval fuzzy probabilities in problems of social interaction based on agents, mainly in decision-making based on game theory. [4, 13]

Acknowledgments. This work was partially supported by the following Brazilian funding agencies: CNPq (Conselho Nacional de Desenvolvimento Científico e Tecnológico), under the Proc. No. 305131/2010-9, 481283/2013-7, 306970/2013-9, 307681/2012-2.

Referências

- [1] D. Adamatti, G. P. Dimuro, H. Coelho (eds.), *Interdisciplinary Applications of Agent-Based Social Simulation and Modeling*, IGI Global, Hershey, 2014.
- [2] W. M. Amaral, *Teoria de jogos nebulosos na resolução de problemas de decisão e conflito de interesses*, Master's thesis, UNICAMP, Campinas (2007).
- [3] T. C. Asmus, *Diffuse probabilities in Bayesian games*, Master's thesis, Universidade Federal do Rio Grande, Rio Grande, Brazil, (adviser: G. P. Dimuro, in Portuguese) (2013).
- [4] T. C. Asmus, G. P. Dimuro, Uma abordagem para jogos Bayesianos fuzzy com base na probabilidade fuzzy de Buckley, *Revista de Informática Teórica e Aplicada – RITA* 20 (1) (2012) 180–215, (in Portuguese).
- [5] T. C. Asmus, G. P. Dimuro, A total order for symmetric triangular (interval) fuzzy numbers, *Mathware & Soft Computing Magazine* 20 (1) (2013) 76–114.
- [6] J. J. Buckley, *Fuzzy Probabilities: New Approach and Applications*, Springer, Berlin, 2005.
- [7] J. J. Buckley, E. Eslami, Uncertain probabilities I: the discrete case, *Soft Computing - A Fusion of Foundations, Methodologies and Applications* 7 (8) (2003) 500–505.
- [8] C. G. Costa, *Probabilidades imprecisas: Intervalar, fuzzy e fuzzy intuicionista*, Master's thesis, Universidade Federal do Rio Grande do Norte - UFRN, Natal (2012).
- [9] G. P. Dimuro, On interval fuzzy numbers, in: *Post-Proceedings of the Workshop-School on Theoretical Computer Science (WEIT 2011)*, IEEE, Los Alamitos, 2011, pp. 3–8.
- [10] G. P. Dimuro, On interval fuzzy numbers, in: *2011 Workshop-School of Theoretical Computer Science (WEIT)*, IEEE, Los Alamitos, 2011, pp. 3–8.
- [11] G. Parente Farias, G. Pereira Dimuro, G. Dimuro Peter, E. de Manuel Jerez, A BDI-fuzzy agent model for exchanges of non-economic services based on the social exchange theory, in: *Intelligent Systems (BRACIS), 2013 Brazilian Conference on*, Los Alamitos, 2013, pp. 26–32.
- [12] F. C. P. Santos, T. F. Rodrigues, H. Donancio, G. Dimuro, D. Adamatti, G. P. Dimuro, E. Manuel Jerez, Analyzing the problem of the modeling of periodic normalized behaviors in multiagent-based simulation of social systems: The case of the san jerónimo vegetable garden of seville, spain, in: B. Kaminski, G. Koloch (eds.), *Advances in Social Simulation*, vol. 229 of *Advances in Intelligent Systems and Computing*, Springer, Berlin, 2014, pp. 61–72.
- [13] Y. Shoham, K. Leyton-Brown, *Multiagent Systems: Algorithmic, Game-Theoretic, and Logical Foundations*, Cambridge University Press, Cambridge, 2009.
- [14] J. Sichman, A. R. Costa, D. Adamatti, G. Dimuro, P. B. F. Buarque, An overview of social simulation research in Brazil, in: *Proceedings of 2012 Third Brazilian Workshop on Social Simulation (BWSS)*, IEEE, Los Alamitos, 2012, pp. 18–22.
- [15] H. Wu, M. Parlar, Games with incomplete information: A simplified exposition with inventory management applications, *International Journal of Production Economics* 133 (2) (2011) 562 – 577.
- [16] L. A. Zadeh, Fuzzy sets, *Information and Control* 8 (3) (1965) 338–353.
- [17] L. A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning - I, *Information Sciences* 8 (3) (1975) 199–249.
- [18] L. A. Zadeh, Fuzzy probabilities, *Information Processing and Management* 20 (3) (1984) 363–372.