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Mixed Virtual Element-Based Numerical Schemes for Nonlinear Problems in Porous Media Flow

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Abstract. In this talk, we present mixed virtual element-based formulations for some nonlinear problems in porous media flow. The aim of this work is to demonstrate the capacity of these numerical schemes to approximate the variables of interest adequately. In particular, we examine the Brinkman and Navier-Stokes-Brinkman flows. The systems are formulated in terms of a pseudostress tensor and the use of Lagrange multipliers. The well-posedness of the associated augmented formulation, along with a priori error bounds for the discrete scheme, has both been established. Finally, we provide some numerical results that confirm the theoretical results.

Keywords. Virtual Elements, Post-processing, Flow in Porous Medium

1 Introduction

We give some notation to be used along the paper, including those already employed above. Firstly, denoting by $\mathbb{I}$ the identity matrix of $\mathrm{R}^{2\times 2}$, and given $\boldsymbol{\tau} := (\tau_{ij})$, $\boldsymbol{\zeta} := (\zeta_{ij}) \in \mathrm{R}^{2\times 2}$, we write as usual $\mathrm{tr}(\boldsymbol{\tau}) := \sum_{i=1}^{2} \tau_{ii}$, $\quad \boldsymbol{\tau}^{\mathbf{d}} := \boldsymbol{\tau} - \frac{1}{2}\mathrm{tr}(\boldsymbol{\tau})\,\mathbb{I}$, $\quad$ and $\quad \boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{2} \tau_{ij}\zeta_{ij}$, which corresponds, respectively, to the trace and the deviator tensor of $\boldsymbol{\tau}$, and to the tensorial product between $\boldsymbol{\tau}$ and $\boldsymbol{\zeta}$. Furthermore, given a generic scalar functional space M, we let $\mathbf{M}$ and $\mathbb{M}$ be its vector and tensorial counterparts, respectively, whose norms and seminorms are denoted exactly as those of M. On the other hand, letting $\mathbf{div}$ (resp. $\mathbf{rot}$) be the usual divergence operator div (resp. rotational operator rot) acting along the rows of a given tensor, we recall that the space $H := \mathbb{H}_0(\mathbf{div};\mathcal{O}) := \Big\{\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div};\mathcal{O}) : \int_{\mathcal{O}} \mathrm{tr}\,\boldsymbol{\tau} = 0\Big\}$, and $X := \mathbb{L}^2_{\mathrm{tr}}(\Omega) := \big\{\mathbf{s} \in \mathbb{L}^2(\Omega) : \quad \mathrm{tr}\,\mathbf{s} = 0\big\}$, equipped with the usual norm are Hilbert spaces and recall that holds the decomposition $\mathbb{H}(\mathbf{div};\mathcal{O}) = \mathbb{H}_0(\mathbf{div};\mathcal{O}) \oplus \mathrm{R}\,\mathbb{I}$. Now, we introduce suitable virtual element subspaces for V and H, together to their respective approximation properties. To this end, we will assume the basic assumptions on meshes that are standard in this context (cf. [2]), that is, given $\{\mathcal{T}_h\}_{h>0}$ a family of decompositions of Ω in polygonal elements K, and given a particular $K \in \mathcal{T}_h$, we denote its barycenter, diameter, and number of edges by $\mathbf{x}_K$, h_K, and d_K, respectively, and define, as usual, $h := \max\{h_K : K \in \mathcal{T}_h\}$. In addition, we assume that there exists a constant $C_{\mathcal{T}} > 0$ such that for each decomposition $\mathcal{T}_h$ and for each $K \in \mathcal{T}_h$ there hold:

a) the ratio between the shortest edge and the diameter h_K of K is bigger than $C_{\mathcal{T}}$, and

b) K is star-shaped with respect to a ball B of radius $C_{\mathcal{T}} h_K$ and center $\mathbf{x}_B \in K$.

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In the following, we denote by $\Pi_k^{\mathcal{O}} : \mathrm{L}^2(\mathcal{O}) \to \mathrm{P}_k(\mathcal{O})$ the $\mathrm{L}^2(\mathcal{O})$-orthogonal projection onto the space $\mathrm{P}_k(\mathcal{O})$, for any $\mathcal{O} \subseteq \mathrm{R}^2$ and $k \geq 0$. In this sense, $\mathcal{O}$ can be a line segment or polygon of $\mathcal{T}_h$. In addition, we will make use of a tensorial version of the aforementioned projector, which is denoted by $\mathbf{\Pi}_k^0$. Given $K \in \mathcal{T}_h$ and an integer $k \geq 0$, we first let $\mathcal{R}_k^K : \mathrm{H}^1(K) \to \mathrm{P}_{k+1}(K)$ be the projection operator defined for each $\psi \in \mathrm{H}^1(K)$ as the unique polynomial $\mathcal{R}_k^K(\psi) \in \mathrm{P}_{k+1}(K)$ satisfying (cf. [1])

$$\begin{aligned} \int_K \nabla \mathcal{R}_k^K(\psi) \cdot \nabla q &= \int_K \nabla \psi \cdot \nabla q \qquad \forall\, q \in \mathrm{P}_{k+1}(K)\,, \\ \int_{\partial K} \mathcal{R}_k^K(\psi) &= \int_{\partial K} \psi\,. \end{aligned} \tag{1}$$

Furthermore, we now consider the finite-dimensional subspace of $\mathrm{C}(\partial K)$ given by

$$\mathrm{B}_k(\partial K) := \Big\{ \psi \in \mathrm{C}(\partial K): \quad \psi|_e \in \mathrm{P}_{k+1}(e)\,, \quad \forall \text{ edge } e \subseteq \partial K \Big\}, \tag{2}$$

define the following local virtual element space (see, e.g. [1])

$$\begin{aligned} V_h^K := \Big\{ \psi \in \mathrm{H}^1(K): \quad &\psi|_{\partial K} \in \mathrm{B}_k(\partial K)\,, \quad \Delta\,\psi \in \mathrm{P}_{k+1}(K)\,, \\ &\text{and} \quad \int_K \big\{ \mathcal{R}_k^K(\psi) - \psi \big\}\, q \,=\, 0 \qquad \forall\, q \in \widetilde{\mathcal{B}}_k(K) \Big\}, \end{aligned} \tag{3}$$

Also, for each $K \in \mathcal{T}_h$ and $k \geq 0$, we introduce the local virtual space H_k^K as follows

$$\begin{aligned} H_k^K := \Big\{ \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}; K) \cap \mathbb{H}(\mathbf{rot}; K): \quad &\boldsymbol{\tau}\mathbf{n}|_e \in \mathbf{P}_k(e) \quad \forall \text{ edge } e \in \partial K, \\ &\mathbf{div}\,(\boldsymbol{\tau}) \in \mathbf{P}_k(K)\,, \quad \text{and} \quad \mathbf{rot}\,(\boldsymbol{\tau}) \in \mathbf{P}_{k-1}(K) \Big\}, \end{aligned} \tag{4}$$

Then, we can define the global space V_k^h and H_k^h, for V and H, respectively, whereas for X we can chosse the global space X_k^h by using of piecewise polynomials. The approximation properties can be found in [4]

2 Flow of Non-Newtonian Fluids in Porous Media

Let Ω be a bounded domain in R^2 with polygonal boundary Γ. Given a volume force $\mathbf{f} \in \mathbf{L}^2(\Omega)$ and a Dirichlet datum $\mathbf{g} \in \mathbf{H}^{1/2}(\Gamma)$, we seek a tensor $\boldsymbol{\sigma}$ (pseudostress), a vector field $\mathbf{u}$ (velocity) and a scalar field p (pressure), such that

$$\begin{aligned} &\boldsymbol{\sigma} = \mu(|\nabla \mathbf{u}|)\nabla \mathbf{u} - p\,\mathbb{I} \quad \text{in} \quad \Omega\,, \quad \alpha \mathbf{u} - \mathbf{div}\,\boldsymbol{\sigma} = \mathbf{f} \quad \text{in} \quad \Omega\,, \\ &\mathrm{div}\,\mathbf{u} = 0 \quad \text{in} \quad \Omega\,, \quad \mathbf{u} = \mathbf{g} \quad \text{on} \quad \Gamma\,, \quad \text{and} \quad \int_\Omega p = 0\,, \end{aligned} \tag{5}$$

where $\mu : \mathrm{R}^+ \to \mathrm{R}$ is the nonlinear kinematic viscosity function of the fluid, and $\alpha > 0$ is a constant approximation of the viscosity divided by the permeability. In what follows, let $\mu_{ij} : \mathrm{R}^{2\times 2} \to \mathrm{R}$ be the mapping given by $\mu_{ij} := \mu(|\mathbf{r}|) r_{ij}$ for each $\mathbf{r} := (r_{ij}) \in \mathrm{R}^{2\times 2}$ and for each $i, j \in \{1, 2\}$. Then, throughout this paper we assume that μ is of class C^1 and that there exist $\gamma_0, \alpha_0 > 0$ such that for each $\mathbf{r} := (r_{ij}), \mathbf{s} := (s_{ij}) \in \mathrm{R}^{2\times 2}$, there hold

$$|\mu_{ij}(\mathbf{r})| \;\leq\; \gamma_0 |\mathbf{r}|, \quad \text{and} \quad \left| \frac{\partial}{\partial r_{kl}} \mu_{ij}(\mathbf{r}) \right| \;\leq\; \gamma_0 \quad \forall\, i, j, k, l \in \{1, 2\}, \tag{6}$$

and

$$\sum_{i,j,k,l=1}^{2} \frac{\partial}{\partial r_{kl}} \mu_{ij}(\mathbf{r}) s_{ij} s_{kl} \;\geq\; \alpha_0 |\mathbf{s}|^2. \tag{7}$$

2.1 Continuous Scheme

Now, as was explained in [3], using the incompressibility condition to eliminate the pressure, and introducing the auxiliary unknown $\mathbf{t} := \nabla\mathbf{u}$ in Ω, we rewrite (5) as follows:

$$\begin{gathered} \mathbf{t} \;=\; \nabla\mathbf{u} \quad \text{in} \quad \Omega, \qquad \boldsymbol{\sigma}^{\mathbf{d}} \;=\; \mu(|\mathbf{t}|)\mathbf{t} \quad \text{in} \quad \Omega\,, \qquad \alpha\mathbf{u} - \mathbf{div}\,\boldsymbol{\sigma} \;=\; \mathbf{f} \quad \text{in} \quad \Omega\,, \\ \operatorname{tr}(\mathbf{t}) \;=\; 0 \quad \text{in} \quad \Omega\,, \qquad \mathbf{u} \;=\; \mathbf{g} \quad \text{on} \quad \Gamma \quad \text{and} \quad \int_\Omega \operatorname{tr}(\boldsymbol{\sigma}) \;=\; 0\,. \end{gathered} \tag{8}$$

We recall that the pressure can be obtained using the formula $p \;=\; -\frac{1}{2}\operatorname{tr}(\boldsymbol{\sigma}) \quad \text{in} \quad \Omega\,$. Note from the fourth and last equation of (8) that $\mathbf{t}$ and $\boldsymbol{\sigma}$ must belong to X and H, respectively. Then, proceeding as in [3, Section 2.2], that is, testing the first two equations of (8) by suitable test functions, integrating by parts, using the Dirichlet conditions for $\mathbf{u}$, the fact that the velocity can be replaced from the third equation of (8) as $\mathbf{u} = \frac{1}{\alpha}\big\{\mathbf{f} + \mathbf{div}(\boldsymbol{\sigma})\big\}$, and adding the following redundant term

$$\kappa \int_\Omega \big\{\boldsymbol{\sigma}^{\mathbf{d}} - \mu(|\mathbf{t}|)\mathbf{t}\big\} : \boldsymbol{\tau}^{\mathbf{d}} \;=\; 0 \qquad \forall\, \boldsymbol{\tau} \in H\,,$$

with κ a positive parameter to be specified later, we arrive at the augmented variational formulation: Find $(\mathbf{t}, \boldsymbol{\sigma}) \in X \times H$ such that

$$[\mathbf{A}(\mathbf{t},\boldsymbol{\sigma}), (\mathbf{s},\boldsymbol{\tau})] \;=\; [\mathbf{F}, (\mathbf{s},\boldsymbol{\tau})] \qquad \forall\, (\mathbf{s},\boldsymbol{\tau}) \in X \times H\,, \tag{9}$$

where $\mathbf{A} : X \times H \to (X \times H)'$ and $\mathbf{F} \in (X \times H)'$ are given by

$$\begin{aligned} [\mathbf{A}(\mathbf{t},\boldsymbol{\sigma}), (\mathbf{s},\boldsymbol{\tau})] \;\; := \;\; & [\mathbb{A}(\mathbf{t}), \mathbf{s} - \kappa\boldsymbol{\tau}^{\mathbf{d}}] \;-\; \int_\Omega \mathbf{s} : \boldsymbol{\sigma}^{\mathbf{d}} \;+\; \int_\Omega \mathbf{t} : \boldsymbol{\tau}^{\mathbf{d}} \;+\; \kappa \int_\Omega \boldsymbol{\sigma}^{\mathbf{d}} : \boldsymbol{\tau}^{\mathbf{d}} \\ & +\; \frac{1}{\alpha} \int_\Omega \mathbf{div}\,\boldsymbol{\sigma} \cdot \mathbf{div}\,\boldsymbol{\tau}\,, \end{aligned} \tag{10}$$

and

$$[\mathbf{F}, (\mathbf{s},\boldsymbol{\tau})] \;:=\; -\frac{1}{\alpha} \int_\Omega \mathbf{f} \cdot \mathbf{div}\,\boldsymbol{\tau} \;+\; \langle \boldsymbol{\tau}\mathbf{n}, \mathbf{g} \rangle\,, \tag{11}$$

respectively, where $\langle \cdot, \cdot \rangle$ stands for the duality pairing between $\mathbf{H}^{-1/2}(\Gamma)$ and $\mathbf{H}^{1/2}(\Gamma)$. In addition, the analysis of the continuous formulation (9) is based on the results of the nonlinear analysis (cf. [3, Section 2.2]). In this way, the well-posedness of the variational formulation (9) is established by the following theorem.

Theorem 2.1. Assume that $\mathbf{f} \in \mathbf{L}^2(\Omega), \mathbf{g} \in \mathbf{H}^{1/2}(\Gamma)$, and that given $\delta \in \Big(0, \frac{2}{\gamma_0}\Big)$, the parameter κ lies in $\Big(0, \frac{2\delta\alpha_0}{\gamma_0}\Big)$. Then, there exists a unique $(\mathbf{t}, \boldsymbol{\sigma}) \in X \times H$ solution of (16). Moreover, there exists a positive constant C, depending only on $\Omega, \alpha_0, \gamma_0, \kappa$ and α, such that

$$\|(\mathbf{t}, \boldsymbol{\sigma})\|_{X \times H} \;\leq\; C\big\{\|\mathbf{f}\|_{0,\Omega} \;+\; \|\mathbf{g}\|_{1/2,\Gamma}\big\}.$$

Proof. See [3, Theorem 2.1]). □

2.2 Discrete Scheme

The mixed virtual element scheme associated with the augmented formulation (9) reads: Find $(\mathbf{t}_h, \boldsymbol{\sigma}_h) \in X_k^h \times H_k^h$ such that

$$[\mathbf{A}_h(\mathbf{t}_h,\boldsymbol{\sigma}_h), (\mathbf{s}_h,\boldsymbol{\tau}_h)] \;=\; [\mathbf{F}, (\mathbf{s}_h,\boldsymbol{\tau}_h)] \qquad \forall\, (\mathbf{s}_h,\boldsymbol{\tau}_h) \in X_k^h \times H_k^h\,. \tag{12}$$

where $\mathbf{A}_h$ is the computable discrete nonlinear operator approximating defined by

$$[\mathbf{A}_h(\mathbf{t}_h,\boldsymbol{\sigma}_h),(\mathbf{s}_h,\boldsymbol{\tau}_h)] := \sum_{K\in\mathcal{T}_h} [\mathbb{A}(\mathbf{t}_h),\mathbf{s}_h - \kappa(\boldsymbol{\Pi}_k^0(\boldsymbol{\tau}_h))^{\mathbf{d}}] - \int_K \mathbf{s}_h : (\boldsymbol{\Pi}_k^0(\boldsymbol{\sigma}_h))^{\mathbf{d}} + \int_K \mathbf{t}_h : (\boldsymbol{\Pi}_k^0(\boldsymbol{\tau}_h))^{\mathbf{d}}$$
$$+ \kappa \int_K (\boldsymbol{\Pi}_k^0(\boldsymbol{\sigma}_h))^{\mathbf{d}} : (\boldsymbol{\Pi}_k^0(\boldsymbol{\tau}_h))^{\mathbf{d}} + \frac{1}{\alpha}\int_K \mathbf{div}\,\boldsymbol{\sigma}_h\cdot\mathbf{div}\,\boldsymbol{\tau}_h + \mathcal{S}^K(\boldsymbol{\sigma}_h - \boldsymbol{\Pi}_k^0(\boldsymbol{\sigma}_h),\boldsymbol{\tau}_h - \boldsymbol{\Pi}_k^0(\boldsymbol{\tau}_h)), \tag{13}$$

where $\mathcal{S}^K : H_k^K \times H_k^K \to \mathrm{R}$ is any symmetric and positive bilinear form verifying (see [3])

$$\widehat{c}_0\|\boldsymbol{\zeta}\|_{0,K}^2 \le \mathcal{S}^K(\boldsymbol{\zeta},\boldsymbol{\zeta}) \le \widehat{c}_1\|\boldsymbol{\zeta}\|_{0,K}^2 \qquad \forall\,\boldsymbol{\zeta}\in H_k^K, \tag{14}$$

with constants $\widehat{c}_0, \widehat{c}_1 > 0$ depending only on $C_{\mathcal{T}}$. The unique solvability and stability of the actual Galerkin scheme (12) is established now

Theorem 2.2. *Assume that given $\delta \in \left(0, \frac{2}{\gamma_0}\right)$, the parameter κ lies in $\left(0, \frac{2\delta\alpha_0}{\gamma_0}\right)$. Then, there exists a unique $(\mathbf{t}_h,\boldsymbol{\sigma}_h) \in X_k^h \times H_k^h$ solution of (12), and there exists a positive constant C, independent of h, such that*

$$\|(\mathbf{t}_h,\boldsymbol{\sigma}_h)\|_{X\times H} \le C\left\{\|\mathbf{f}\|_{0,\Omega} + \|\mathbf{g}\|_{1/2,\Gamma}\right\}.$$

Proof. [3, Theorem 3.1]). □

3 Models with Nonselenoidal Velocity

We consider the Navier-Stokes-Brinkman problem with nonsolenoidal velocity, which is given by the following system of partial differential equations

$$-\mu\,\Delta\,\mathbf{u} + (\nabla\,\mathbf{u})\,\mathbf{u} + \nabla\,p - \mathbb{K}\,\mathbf{u} = \mathbf{f} \quad \text{in} \quad \Omega\,, \quad \operatorname{div}\mathbf{u} = g \quad \text{in} \quad \Omega\,,$$
$$\mathbf{u} = \mathbf{u}_D \quad \text{on} \quad \Gamma \quad \text{and} \quad \int_\Omega p = 0\,, \tag{15}$$

where the unknowns are the velocity $\mathbf{u}$ and the pressure p of a fluid occupying the region Ω. In addition, $\mu > 0$ is the dynamic fluid viscosity, $\mathbb{K}$ is a tensor that represents the permeability of the porous medium, $\mathbf{f}$ and g are given data that represent external body forces and sources and/or skins in Ω, respectively, and $\mathbf{u}_D$ are boundary data.

3.1 Continuous Scheme

Recalling the variational formulation proposed in [6, Section 2] and using minor changes caused by the tensor $\mathbb{K}$, we arrive at the following continuous scheme (with positive parameters κ_1 and κ_2 to be specified later): Find $(\boldsymbol{\sigma},\mathbf{u}) \in H \times V$ such that

$$\mathcal{A}\,((\boldsymbol{\sigma},\mathbf{u}),(\boldsymbol{\tau},\mathbf{v})) + \mathcal{B}\,(\mathbf{u}\,;(\boldsymbol{\sigma},\mathbf{u}),(\boldsymbol{\tau},\mathbf{v})) = \mathcal{F}\,(\boldsymbol{\tau},\mathbf{v})\,, \tag{16}$$

for all $(\boldsymbol{\tau},\mathbf{v}) \in H \times V$, where the forms $\mathcal{A}$ and $\mathcal{B}$ are defined, respectively, as

$$\mathcal{A}\,((\boldsymbol{\sigma},\mathbf{u}),(\boldsymbol{\tau},\mathbf{v})) := \int_\Omega \boldsymbol{\sigma}^{\mathbf{d}} : \boldsymbol{\tau}^{\mathbf{d}} + \mu\int_\Omega g\mathbb{K}^{-1}\,\mathbf{u}\cdot\mathbf{div}\,\boldsymbol{\tau} + \mu\int_\Omega \mathbb{K}^{-1}\mathbf{div}\,\boldsymbol{\sigma}\cdot\mathbf{div}\,\boldsymbol{\tau}$$
$$+ \kappa_1\,\mu\int_\Omega \nabla\,\mathbf{u} : \nabla\,\mathbf{v} - \kappa_1\int_\Omega \boldsymbol{\sigma}^{\mathbf{d}} : \nabla\mathbf{v} + \kappa_2\int_\Gamma \mathbf{u}\cdot\mathbf{v}\,,$$

and

$$\mathcal{B}(\mathbf{u}\,;(\boldsymbol{\sigma},\mathbf{u}),(\boldsymbol{\tau},\mathbf{v})) \;:=\; \int_\Omega (\mathbf{u}\otimes\mathbf{u})^{\mathbf{d}} : \left\{\boldsymbol{\tau} - \kappa_1\,\nabla\,\mathbf{v}\right\}$$

In turn, the bounded functional $\mathcal{F} : H \times V \to \mathrm{R}$ is given by

$$\mathcal{F}(\boldsymbol{\tau},\mathbf{v}) := -\mu\int_\Omega \mathbb{K}\mathbf{f}\cdot\mathbf{div}\,\boldsymbol{\tau} - \frac{\mu}{2}\int_\Omega g\,\mathrm{tr}\,\boldsymbol{\tau} + \mu\langle\boldsymbol{\tau}\mathbf{n},\mathbf{u}_D\rangle + \frac{\kappa_1\mu}{2}\int_\Omega g\,\mathrm{div}\,\mathbf{v} + \kappa_2\int_\Gamma \mathbf{u}_D\cdot\mathbf{v}.$$

where $\langle\cdot,\cdot\rangle$ stands for the duality pairing between $\mathbf{H}^{-1/2}(\Gamma)$ and $\mathbf{H}^{1/2}(\Gamma)$. Now, the analysis of the continuous formulation (16) is based on a fixed point strategy combining the classical Banach Theorem and the Lax-Milgram Theorem. Finally, the following result establishes the well-posedness of the scheme (16).

Theorem 3.1. *Assume appropriate values for κ_1 and κ_2. In addition, suppose that the data $\mathbf{f}$, g and $\mathbf{u}_D$ are sufficiently small. Then, there exists a unique $(\boldsymbol{\sigma},\mathbf{u})\in\mathbf{H}$ solution of (16) that holds*

$$\|(\boldsymbol{\sigma},\mathbf{u})\| \;\leq\; C\left\{\|\mathbf{f}\|_{0,\Omega} + \|g\|_{0,\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\mathbf{u}_D\|_{0,\Gamma}\right\}.$$

Proof. It is a light modification of [6, Theorem 3.4]. □

3.2 Discrete Scheme

Inspired by [4, 5] we get the following discrete scheme : Find $(\boldsymbol{\sigma}_h,\mathbf{u}_h)\in H_h\times V_h$ such that

$$\mathcal{A}_h\left((\boldsymbol{\sigma}_h,\mathbf{u}_h),(\boldsymbol{\tau}_h,\mathbf{v}_h)\right) + \mathcal{B}_h\left(\mathbf{u}_h\,;(\boldsymbol{\sigma}_h,\mathbf{u}_h),(\boldsymbol{\tau}_h,\mathbf{v}_h)\right) \;=\; \mathcal{F}_h\left(\boldsymbol{\tau}_h,\mathbf{v}_h\right), \tag{17}$$

for all $(\boldsymbol{\tau}_h,\mathbf{v}_h)\in H_h\times V_h$, where the forms $\mathcal{A}_h$ and $\mathcal{B}_h$ are defined, respectively, as

$$\begin{aligned}\mathcal{A}_h\left((\boldsymbol{\sigma}_h,\mathbf{u}_h),(\boldsymbol{\tau}_h,\mathbf{v}_h)\right) := \quad & \sum_{K\in\mathcal{T}_h}\int_K \boldsymbol{\Pi}_k^0(\boldsymbol{\sigma}_h)^{\mathbf{d}} : \boldsymbol{\Pi}_k^0(\boldsymbol{\tau}_h)^{\mathbf{d}} + \mathcal{S}^{K,\mathbf{d}}(\boldsymbol{\sigma}_h - \boldsymbol{\Pi}_k^0(\boldsymbol{\sigma}_h), \boldsymbol{\tau}_h - \boldsymbol{\Pi}_k^0(\boldsymbol{\tau}_h))\\ & +\kappa_1\,\mu\int_K \nabla\Pi_k^\nabla(\mathbf{u}_h) : \nabla\,\Pi_k^\nabla(\mathbf{v}_h) + \mathcal{S}^{K,\nabla}(\mathbf{u}_h - \Pi_k^\nabla(\mathbf{u}_h), \mathbf{v}_h - \Pi_k^\nabla(\mathbf{v}_h))\\ & +\mu\int_K g\mathbb{K}^{-1}\,\Pi_k^0(\mathbf{u}_h)\cdot\mathbf{div}\,\boldsymbol{\tau}_h + \mu\int_K \mathbb{K}^{-1}\mathbf{div}\,\boldsymbol{\sigma}_h\cdot\mathbf{div}\,\boldsymbol{\tau}_h\\ & -\kappa_1\int_K \boldsymbol{\Pi}_k^0(\boldsymbol{\sigma}_h)^{\mathbf{d}} : \boldsymbol{\Pi}_k^0(\nabla\mathbf{v}_h) + \kappa_2\int_{\partial K}\mathbf{u}_h\cdot\mathbf{v}_h\,,\end{aligned}$$

and

$$\mathcal{B}_h(\mathbf{u}_h\,;(\boldsymbol{\sigma}_h,\mathbf{u}_h),(\boldsymbol{\tau}_h,\mathbf{v}_h)) \;:=\; \sum_{K\in\mathcal{T}_h}\int_K (\Pi_k^0(\mathbf{u}_h)\otimes\Pi_k^0(\mathbf{u}_h))^{\mathbf{d}} : \left\{\boldsymbol{\Pi}_k^0(\boldsymbol{\tau}_h) - \kappa_1\,\boldsymbol{\Pi}_k^0(\nabla\,\mathbf{v}_h)\right\}.$$

In turn, the bounded functional $\mathcal{F}_h : H_h\times V_h \to \mathrm{R}$ is given by

$$\begin{aligned}\mathcal{F}_h(\boldsymbol{\tau}_h,\mathbf{v}_h) \quad := \quad & \sum_{K\in\mathcal{T}_h} -\mu\int_K \mathbb{K}\Pi_k^0(\mathbf{f})\cdot\mathbf{div}\,\boldsymbol{\tau}_h - \frac{\mu}{2}\int_K g\,\mathrm{tr}\,\boldsymbol{\tau}_h + \mu\langle\boldsymbol{\tau}_h\mathbf{n},\mathbf{u}_D\rangle\\ & +\frac{\kappa_1\mu}{2}\int_K g\,\mathrm{div}\,\mathbf{v}_h + \kappa_2\int_{\partial K}\mathbf{u}_D\cdot\mathbf{v}_h.\end{aligned}$$

Now, the analysis of (16) again is based on a fixed point strategy combining the classical Banach Theorem and the Lax-Milgram Theorem, but now combined with strategies taken of [4, 5]. Finally, the following result establishes the well-posedness of the scheme (17).

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Theorem 3.2. *Let* $(\boldsymbol{\sigma}, \mathbf{u}) \in H \times V$ *and* $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in H_k^h \times V_k^h$ *be the unique solutions of the continuous and discrete schemes* (16) *and* (17)*, respectively. Assume that for integers* $r \in [1, k+1]$*, and* $s \in [2, k+1]$*, there hold* $\boldsymbol{\sigma}|_K \in \mathbb{H}^r(K)$*,* $\mathbf{div}\,\boldsymbol{\sigma}|_K \in \mathbf{H}^r(K)$*,* $\mathbf{u}|_K \in \mathbf{H}^s(K)$*,and* $g|_K = \operatorname{div} \mathbf{u}|_K \in \mathrm{H}^s(K)$*, for each* $K \in \mathcal{T}_h$*. Then, there exists a positive constant* C*, independent of* h*, such that*

$$\|(\boldsymbol{\sigma}, \mathbf{u}) - (\boldsymbol{\sigma}_h, \mathbf{u}_h)\| \leq C\, h^{\min\{r, s-1, m-1\}} \left\{ \sum_{K \in \mathcal{T}_h} \left(|\boldsymbol{\sigma}|_{r,K}^2 + |\mathbf{div}\,\boldsymbol{\sigma}|_{r,K}^2 + |\mathbf{u}|_{s,K}^2 + |\operatorname{div} \mathbf{u}|_{s,K}^2 \right) \right\}$$
$$+ C\, h^{s-1} \left\{ \sum_{K \in \mathcal{T}_h} |\mathbf{u}|_{s-1,4,K}^4 + |\operatorname{div} \mathbf{u}|_{s-1,4,K}^4 \right\}^{1/4}.$$

Proof. This follows from Strang-type estimates and the approximation properties of the virtual spaces involved (See [4, Theorem 5.3]) . □

4 Numerical Results

We present two numerical experiments illustrating the performance of the augmented mixed virtual element scheme (12). In Example 1 we take the unit square $\Omega := (0,1)^2$, set $\alpha = 1$, and consider the nonlinear viscosity μ given by the Carreau law, that is $\mu(s) := 2 + (1 + s^2)^{-1/6} \quad \forall\, s \geq 0$. In addition, we choose the data $\mathbf{f}$ and $\mathbf{g}$ so that the exact solution is given by $\mathbf{u}(\mathbf{x}) := \begin{pmatrix} -\cos(\pi x_1)\sin(\pi x_2) \\ \sin(\pi x_1)\cos(\pi x_2) \end{pmatrix}$ and $p(\mathbf{x}) := x_1^2 - x_2^2$ for all $\mathbf{x} := (x_1, x_2)^{\mathbf{t}} \in \Omega$. In Example 2 we take the L-shaped domain $\Omega := (-1,-1)^2 \setminus [0,1]^2$, set again $\alpha = 1$, and consider the nonlinear viscosity given by $\mu(s) := \frac{1}{2} + \frac{1}{2}(1 + s^2)^{-1/4} \quad \forall\, s \geq 0$. Then, we choose the data $\mathbf{f}$ and $\mathbf{g}$ so that the exact solution is given by $\mathbf{u}(\mathbf{x}) := \begin{pmatrix} (1 + x_1 - \exp(x_1))(1 - \cos(x_2)) \\ (1 - \exp(x_1))(\sin(x_2) - x_2) \end{pmatrix}$ and $p(\mathbf{x}) := (x_1^2 + x_2^2)^{1/3} - p_0$ for all $\mathbf{x} := (x_1, x_2)^{\mathbf{t}} \in \Omega$, where $p_0 \in \mathrm{R}$ is such that $\int_\Omega p = 0$. Note in this example that the partial derivatives of p, and hence, in particular $\mathbf{div}\,\boldsymbol{\sigma}$, are singular at the origin. More precisely, because of the power 1/3, there holds $\boldsymbol{\sigma} \in \mathbb{H}^{5/3-\epsilon}(\Omega)$ and $\mathbf{div}\,\boldsymbol{\sigma} \in \mathbf{H}^{2/3-\epsilon}(\Omega)$ for each $\epsilon > 0$.

Table 1: Example 1, history of convergence using quadrilaterals.

k	h	N	$\mathtt{e}(\mathbf{t})$	$\mathtt{r}(\mathbf{t})$	$\mathtt{e}_0(\boldsymbol{\sigma})$	$\mathtt{r}_0(\boldsymbol{\sigma})$	$\mathtt{e}_{\mathrm{div}}(\boldsymbol{\sigma})$	$\mathtt{r}_{\mathrm{div}}(\boldsymbol{\sigma})$	$\mathtt{e}(\mathbf{u})$	$\mathtt{r}(\mathbf{u})$	$\mathtt{e}(p)$	$\mathtt{r}(p)$	$\mathtt{e}(\boldsymbol{\sigma}^\star)$	$\mathtt{r}(\boldsymbol{\sigma}^\star)$
	0.0461	8716	1.18e-1	−−	3.16e-1	−−	3.79e+1	−−	2.73e-2	−−	2.84e-2	−−	1.42e-0	−−
	0.0359	14356	9.16e-2	1.00	2.46e-1	1.00	3.79e+1	0.00	2.09e-2	1.06	2.16e-2	1.08	1.10e-0	1.00
0	0.0183	54561	4.68e-2	1.00	1.26e-1	1.00	3.79e+1	0.00	1.05e-2	1.03	1.06e-2	1.06	5.64e-1	1.00
	0.0135	101281	3.43e-2	1.00	9.21e-2	1.00	3.79e+1	0.00	7.67e-3	1.01	7.80e-3	1.00	4.14e-1	1.00
	0.0101	179841	2.58e-2	1.00	6.90e-2	1.00	3.79e+1	0.00	5.74e-3	1.01	5.84e-3	1.01	3.10e-1	1.00
	0.0461	28456	2.72e-3	−−	7.56e-3	−−	1.40e-0	−−	5.81e-4	−−	1.41e-3	−−	3.56e-2	−−
	0.0359	46936	1.64e-3	2.02	4.53e-3	2.04	1.09e-0	1.00	3.50e-4	2.02	7.79e-4	2.35	2.15e-2	2.00
1	0.0183	178817	4.25e-4	2.01	1.16e-3	2.03	5.57e-1	1.00	9.12e-5	2.01	1.68e-4	2.29	5.64e-3	1.99
	0.0135	332161	2.28e-4	2.01	6.22e-4	2.02	4.09e-1	1.00	4.90e-5	2.00	8.38e-5	2.23	3.04e-3	2.00
	0.0101	590081	1.28e-4	2.01	3.48e-4	2.01	3.07e-1	1.00	2.75e-5	2.00	4.46e-5	2.19	1.71e-3	2.00
	0.0461	56771	4.28e-5	−−	1.39e-4	−−	3.57e-2	−−	8.52e-6	−−	3.33e-5	−−	1.89e-3	−−
	0.0359	93691	2.00e-5	3.03	6.46e-5	3.06	2.16e-2	2.00	4.00e-6	3.00	1.47e-5	3.24	9.09e-4	2.92
2	0.0183	357281	2.64e-6	3.02	8.56e-6	3.02	5.68e-3	1.99	5.35e-7	3.00	1.85e-6	3.09	1.24e-4	2.97
	0.0135	663841	1.04e-6	3.01	3.36e-6	3.01	3.06e-3	1.99	2.11e-7	3.00	7.07e-7	3.10	4.90e-5	3.00
	0.0101	1179521	4.36e-7	3.02	1.41e-6	3.02	1.72e-3	2.00	8.89e-8	3.00	2.92e-7	3.07	2.07e-5	3.00

Table 2: Example 2, history of convergence using hexagons.

k	h	N	$\mathtt{e}(\mathbf{t})$	$\mathtt{r}(\mathbf{t})$	$\mathtt{e}_0(\boldsymbol{\sigma})$	$\mathtt{r}_0(\boldsymbol{\sigma})$	$\mathtt{e}_{\mathrm{div}}(\boldsymbol{\sigma})$	$\mathtt{r}_{\mathrm{div}}(\boldsymbol{\sigma})$	$\mathtt{e}(\mathbf{u})$	$\mathtt{r}(\mathbf{u})$	$\mathtt{e}(p)$	$\mathtt{r}(p)$	$\mathtt{e}(\boldsymbol{\sigma}^\star)$	$\mathtt{r}(\boldsymbol{\sigma}^\star)$
	0.0866	8382	2.26e-2	−−	4.02e-2	−−	1.56e-0	−−	6.62e-3	−−	2.41e-2	−−	1.18e-1	−−
	0.0462	28668	1.20e-2	1.01	2.10e-2	1.03	1.56e-0	0.00	3.49e-3	1.02	1.25e-2	1.04	7.73e-2	0.68
0	0.0315	61050	8.13e-3	1.02	1.42e-2	1.02	1.56e-0	0.00	2.38e-3	1.00	8.48e-3	1.02	5.98e-2	0.67
	0.0247	98868	6.38e-3	1.00	1.11e-2	1.02	1.56e-0	0.00	1.87e-3	1.00	6.63e-3	1.02	5.16e-2	0.61
	0.0204	145758	5.25e-3	1.01	9.16e-3	1.01	1.56e-0	0.00	1.54e-3	0.99	5.45e-3	1.01	4.60e-2	0.58
	0.0866	25142	5.93e-4	−−	1.50e-3	−−	1.15e-1	−−	1.86e-4	−−	9.84e-4	−−	4.87e-2	−−
	0.0462	86000	1.84e-4	1.86	5.45e-4	1.61	7.66e-2	0.65	5.38e-5	1.97	3.65e-4	1.58	3.24e-2	0.65
1	0.0315	183146	9.66e-5	1.68	2.96e-4	1.60	5.98e-2	0.65	2.52e-5	1.98	1.99e-4	1.59	2.63e-2	0.54
	0.0247	296600	6.30e-5	1.77	2.02e-4	1.59	5.16e-2	0.61	1.55e-5	2.01	1.36e-4	1.57	2.26e.2	0.62
	0.0204	437270	4.48e-5	1.76	1.49e-4	1.55	4.52e-2	0.68	1.06e-5	1.99	1.01e-4	1.53	2.00e-2	0.63
	0.0866	48419	9.62e-5	−−	4.32e-4	−−	5.08e-2	−−	4.09e-6	−−	2.98e-4	−−	3.35e-2	−−
	0.0462	165627	3.32e-5	1.69	1.58e-4	1.60	3.41e-2	0.63	7.64e-7	2.67	1.09e-4	1.59	2.31e-2	0.59
2	0.0315	352723	2.01e-5	1.31	9.18e-5	1.42	2.73e-2	0.58	2.65e-7	2.77	6.33e-5	1.43	1.85e-2	0.58
	0.0247	571227	1.24e-5	2.00	6.32e-5	1.55	2.36e-2	0.59	1.50e-7	2.34	4.38e-5	1.53	1.61e-2	0.56
	0.0204	842174	9.85e-6	1.19	4.76e-5	1.46	2.10e-2	0.61	8.41e-8	3.00	3.29e-5	1.47	1.44e-2	0.58

5 Conclusions

This work highlights the effectiveness of mixed virtual element methods for modeling porous media, particularly in nonlinear settings. Their flexible structure supports adaptive strategies that enhance both accuracy and computational efficiency by locally refining the mesh or increasing the approximation order based on solution features, as demonstrated in [7].

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