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All Direct Product $C_5 \times K_n$ Graphs Are Type 1

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Abstract. A *k-total coloring* of a graph G is an assignment of k colors to the elements (vertices and edges) of G so that adjacent or incident elements have different colors. The total chromatic number is the smallest integer k for which G has a k-total coloring. The well known Total Coloring Conjecture states that the total chromatic number of a graph is either $\Delta(G) + 1$ (called Type 1) or $\Delta(G) + 2$ (called Type 2), where $\Delta(G)$ is the maximum degree of G. In this paper, we establish that all the direct product $C_5 \times K_n$ graphs are Type 1, when n is odd and not a multiple of 5, providing evidence for the conjecture that all $C_m \times K_n$ graphs are Type 1.

Keywords. Graph Theory, Total Coloring, Direct Product

1 Introduction

Let G be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. A *k-total coloring* of a graph G is an assignment of k colors to the elements (vertices and edges) of G so that adjacent or incident elements have different colors. The *total chromatic number*, denoted by $\chi_T(G)$, is the smallest integer k for which G has a k-total coloring. Clearly, $\chi_T(G) \geq \Delta(G) + 1$ and the *Total Coloring Conjecture* (TCC), posed independently by Vizing [7] and Behzad [2], states that $\chi_T(G) \leq \Delta(G) + 2$, where $\Delta(G)$ is the maximum degree of G. Graphs with $\chi_T(G) = \Delta(G) + 1$ are said to be *Type* 1 and graphs with $\chi_T(G) = \Delta(G) + 2$ are said to be *Type* 2. The TCC has been verified in restricted cases, such as cubic graphs [6], but has not been settled for all regular graphs for more than fifty years.

We denote an undirected edge $e \in E(G)$ whose ends are u and v by uv. The *direct product* (also called *tensor product* or *categorical product*) of two graphs G and H is a graph denoted by $G \times H$, whose vertex set is the Cartesian product $V(G) \times V(H)$, for which vertices (u, v) and (u', v') are adjacent if and only if $uu' \in E(G)$ and $vv' \in E(H)$. The maximum degree of $G \times H$ is $\Delta(G \times H) = \Delta(G) \cdot \Delta(H)$, and $G \times H$ is regular if and only if both G and H are regular graphs. Let $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ be two graphs on the same vertex set V and where $E_1 \cap E_2 = \emptyset$,

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and denote by $\bigoplus_{i=1}^{2} G_i$ the direct sum graph $G = (V, E_1 \cup E_2)$ of graphs G_1 and G_2. In this work, given two graphs G and H, we use the well-known property that the direct product is distributive over edge-disjoint union of graphs, that is, if $G = \bigoplus_{i=1}^{t} G_i$, where G_i are edge-disjoint subgraphs of G and $E(G) = E(G_1) \cup E(G_2) \cup \ldots \cup E(G_t)$, then $H \times G = \bigoplus_{i=1}^{t} (H \times G_i)$.

The complete graph on n vertices is denoted by K_n and the cycle graph on n vertices is denoted by C_n. We proved in [4] that the direct product of two complete graphs $K_m \times K_n$ is Type 1, except for $K_2 \times K_2$. We previously proved that the direct product of two cycles $C_m \times C_n$ is Type 1, except for $C_4 \times C_4$ in [3]. In [4] we started investigating the total coloring on direct product of a cycle with a complete graph, denoted by $C_m \times K_n$. Unlike the previous families, we presented a Type 2 infinite family $C_m \times K_2$ when m is not a multiple of 3, and conjectured that all other cases are Type 1. The remaining unproven cases of this conjecture belong to the family $C_m \times K_n$ when $m, n \geq 5$ are odd. In this work, we focus on one such case by proving that the family $C_5 \times K_n$ is Type 1, when n is odd and not a multiple of 5, contributing to the conjecture presented in [4].

In order to determine the optimal total colorings for all members of the family $C_5 \times K_n$, when n is odd and not multiple of 5, we use the guiding color technique introduced in [4]. Such technique uses a Hamiltonian decomposition together with a color class with specific properties, called guiding color class, that guides how to construct the desired total coloring. This technique can be applied to any graph with a Hamiltonian decomposition. For the family $C_m \times K_n$, when m, n are odd and $gcd(m, n) = 1$, there is a natural Hamiltonian decomposition that is inspired from the one of $K_m \times K_n$ given in [4]. The choice of guiding color class is laborious and combinatorial. Even after successfully assigning such a guiding color to a family of graphs, there are remaining particular graphs to consider independently.

In the following, we provide $(\Delta(C_5 \times K_n) + 1)$-total colorings of $C_5 \times K_n$, for odd number n with $gcd(5, n) = 1$, determining that all theses graphs are Type 1.

2 The Guiding Color Technique

For $C_5 \times K_n$ with $n \geq 3$ an odd number, in Subsection 2.1, we use Walecki's Hamiltonian decomposition of K_n to define suitable Hamiltonian decompositions of $C_5 \times K_n$, when $gcd(5, n) = 1$, that is, when n is not a multiple of 5. In Subsection 2.2, we give an assignment of the guiding color to these graphs. The Hamiltonian decomposition constructed in Subsection 2.1 and the guiding color given in Section 3 define the target $(\Delta(C_5 \times K_n) + 1)$-total coloring.

2.1 Hamiltonian Decompositions

A k-regular graph G has a *Hamiltonian decomposition* (or is *Hamiltonian decomposable*) if its edge set can be partitioned into $\frac{k}{2}$ Hamiltonian cycles when k is an even number, or into $\frac{(k-1)}{2}$ Hamiltonian cycles plus a one factor (or perfect matching) when k is an odd number. Please refer to [1] for a survey on Hamiltonian decompositions.

As the cycle graph C_5 is a unique Hamiltonian cycle, for the sake of simplicity, we consider the sequential order of $C_5 = \langle 0, 1, 2, 3, 4, 0 \rangle$.

Consider the well known Walecki's Hamiltonian decomposition of the complete graph K_n for $n \geq 3$. We shall focus on an odd number n. Let $n = 2w + 1$ and label the vertices of K_n as $0, 1, \ldots, 2w$. Following the notation used in [1], let C_n be the Hamiltonian cycle $\langle 0, 1, 2, 2w, 3, 2w-1, 4, 2w-2, 5, 2w-3, \ldots, w+3, w, w+2, w+1, 0 \rangle$. If σ is the permutation $(0)(1, 2, 3, 4, \ldots, 2w-1, 2w)$, then $\sigma^0(C_n), \sigma^1(C_n), \sigma^2(C_n), \ldots, \sigma^{w-1}(C_n)$ is a Hamiltonian decomposition of K_n. Observe that

$\sigma^0(C_n) = C_n$. We write $K_n = \bigoplus_{i=1}^{w} \sigma^{i-1}(C_n)$. Denote by $\sigma^{i-1}(C_n)_z$, with $z = 0, 1, \ldots, n-1$, the z^{th}-vertex in the cycle $\sigma^{i-1}(C_n)$, and in fact, vertex 0 is always the starting vertex (0^{th}-vertex). Note that for $i \leq w$, the cycle $\sigma^{i-1+w}(C_n)$ is the opposite cycle of $\sigma^{i-1}(C_n)$, that is, $\sigma^{i-1+w}(C_n)_z = \sigma^{i-1}(C_n)_{n-z}$ for all $z \geq 1$.

Consider for instance $n = 7$, to get the Hamiltonian decomposition $K_7 = \bigoplus_{i=1}^{3} \sigma^{i-1}(C_7)$, where $\sigma^0(C_7) = \langle 0, 1, 2, 6, 3, 5, 4, 0\rangle$ and $\sigma^1(C_7) = \langle 0, 2, 3, 1, 4, 6, 5, 0\rangle$ and $\sigma^2(C_7) = \langle 0, 3, 4, 2, 5, 1, 6, 0\rangle$. Note that $\sigma^3(C_7) = \langle 0, 4, 5, 3, 6, 2, 1, 0\rangle$ is the opposite cycle of $\sigma^0(C_7)$, $\sigma^4(C_7) = \langle 0, 5, 6, 4, 1, 3, 2, 0\rangle$ is the opposite cycle of $\sigma^1(C_7)$ and $\sigma^5(C_7) = \langle 0, 6, 1, 5, 2, 4, 3, 0\rangle$ is the opposite cycle of $\sigma^2(C_7)$.

It is well known and not hard to see that the direct product of cycle graphs is Hamiltonian decomposable if and only if at least one of them is an odd cycle [5]. In what follows, for n odd number, we shall use Walecki's Hamiltonian decomposition of the complete graph K_n and the well known distributive property of the direct product to define a Hamiltonian decomposition of $C_5 \times K_n$ and odd number $n \geq 7$ suitable to our target total coloring.

Let $gcd(5, n) = 1$ and $n = 2w + 1$. For $i = 1, \ldots, 2w$, denote by $C(i)$ the cycle on $5n$ vertices $\langle C(i)_z\rangle_{z=0,\ldots,5n}$, where $C(i)_z = (z \bmod 5, \sigma^{i-1}(C_n)_{z \bmod n})$, with $z = 0, \ldots, 5n$, is the z^{th}-vertex of the cycle $C(i)$. Observe that according to the notation for vertex $C(i)_z$, we have $C(i)_0 = C(i)_{5n}$, and the vertex $(0, 0)$ is always the 0^{th}-vertex of $C(i)$. For instance, Figure 1 presents the cycle $C(1)$ using the cycle C_5 and the cycle $\sigma^0(C_7)$ of K_7.

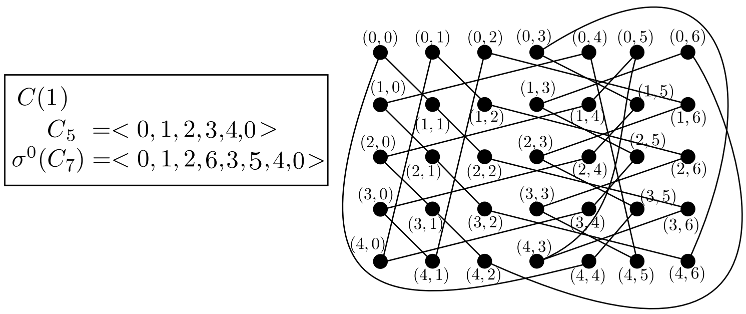

Figure 1: The cycle $C(1)$ in $C_5 \times K_7$. Source: Authors.

We consider next the construction of a Hamiltonian decomposition of $C_5 \times K_n$ according to whether $gcd(m, n) = 1$. Consider $\{C(i) \mid i = 1, \ldots, 2w\}$, a Hamiltonian decomposition of $C_5 \times K_n$. Indeed, consider $C_5 = \langle 0, 1, 2, 3, 4, 0\rangle$ and $K_n = \bigoplus_{i=1}^{w} (\sigma^{i-1}(C_n))$ the Walecki's Hamiltonian decompositions of K_n. As the degree $\Delta(C_5 \times \sigma^{i-1}(C_n)) = 4$, for any $i = 1, 2, \ldots, 2w$, each subgraph $C_5 \times \sigma^{i-1}(C_n)$ of $C_5 \times K_n$ has two Hamiltonian cycles: $C(i)$ and $C(i + w)$, and so, it suffices to consider $C(i)$ for $i = 1, \ldots, 2w$.

For instance, consider $C_5 \times K_7$ in Figure **??**. As $gcd(5, 7) = 1$ we use $C_5 \times K_7 = \bigoplus_{i=1}^{3} (C_5 \times \sigma^{i-1}(C_7))$, the 2 Hamiltonian cycles of the subgraph $C_5 \times \sigma^0(C_7)$ of $C_5 \times K_7$ are $C(1)$ and $C(4)$. Analogously, the 2 Hamiltonian cycles of the subgraph $C_5 \times \sigma^1(C_7)$ of $C_5 \times K_7$ are $C(2)$ and $C(5)$, and the 2 Hamiltonian cycles of the subgraph $C_5 \times \sigma^2(C_7)$ of $C_5 \times K_7$ are $C(3)$ and $C(6)$.

2.2 The Guiding Color: an Example

We are ready to explain how a Type 1 $(2n - 1)$-total coloring of $C_5 \times K_n$ is obtained by considering the Hamiltonian decomposition of $C_5 \times K_n$ into Hamiltonian cycles $C(i)$ defined in

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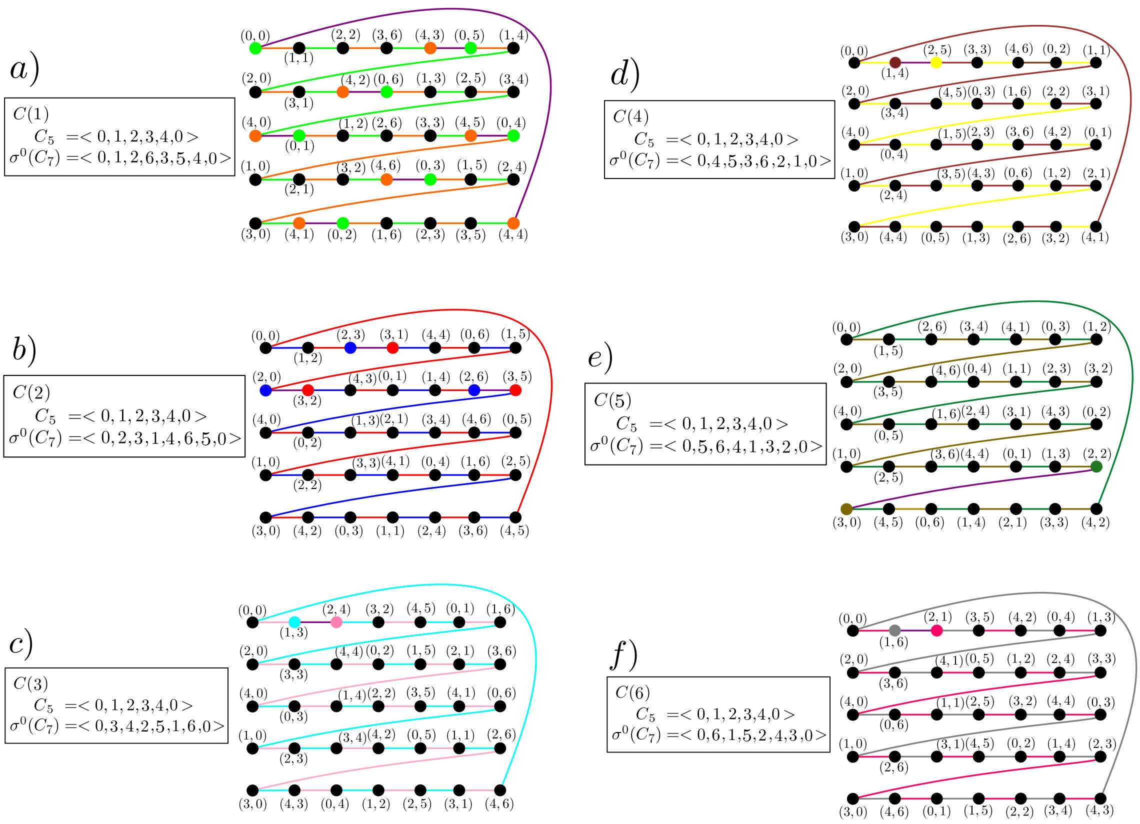

Figure 2: A depiction of $C_5 \times K_7$ partitioned into 6 Hamiltonian cycles. In (a) we have the Hamiltonian cycle $C(1)$ with 3 colors: the edges $(4,3)(0,5)$, $(4,2)(0,6)$, $(4,0)(0,1)$, $(4,5)(0,4)$, $(4,6)(0,3)$, $(4,1)(0,2)$ and $(4,4)(0,0)$ are colored with the guiding purple color; the endpoints of the purple edges and the remaining edges of $C(1)$ are colored with colors orange and green. In (b) we have the Hamiltonian cycle $C(2)$ also colored with 3 colors: the edges $(2,3)(3,1)$, $(2,0)(3,2)$ and $(2,6)(3,5)$ colored with the guiding purple color; the endpoints of the purple edge and the remaining edges of $C(2)$ are colored with colors red and blue. In the remaining 4 Hamiltonian cycles, each of them has one edge with the guiding purple color whose endpoints and the remaining edges of the cycle are colored with additional 2 new colors. Source: Authors.

Subsection 2.1. In a $(2n-1)$-total coloring, each color class is such that each vertex is either inside the color class or is incident to an edge of the color class. We shall choose a guiding color with the additional property that its color class contains one or three edges of each Hamiltonian cycle. Note that each Hamiltonian cycle is an odd cycle and, by Vizing's theorem [7], admits a 3-edge coloring. Thus, for each cycle, we assign two additional colors to the remaining edges of the Hamiltonian cycle and to the endpoints of the edges with the guiding color. Figure **??** presents the 6 colored Hamiltonian cycles of $C_5 \times K_7$, which partitioned the set of edges of the graph.

With suitable choices for the edges of the matching colored by the guiding color, the so far uncolored vertices define an independent set which can be also colored with the guiding color.

In order to obtain a $(2n-1)$-total coloring, we give a table composed by the elements of the guiding color class. We identify the edges of the guiding color on the corresponding Hamiltonian cycle where they belong. If the Hamiltonian cycle contains a unique edge of the guiding color, then its endpoints and the remaining edges of the cycle are easily colored using two additional colors. If the Hamiltonian cycle contains three edges of the guiding color, then we can easily see that their endpoints define two independent sets that can be colored with two colors as also the remaining edges of the cycle.

In Figure 3, for the graph $C_5 \times K_7$, we represent a table and a subgraph highlighting all elements (edges and vertices) colored by the guiding color and the colored vertices of Figure **??**. We can identify which of the six Hamiltonian cycles contains which highlighted edges by observing the colors of their endpoints. In Figure **??**(a), the fourteen endpoints of the seven edges colored with the guiding color (purple) in $C(1)$ are the seven vertices $(4, i)$, for $i = 0, \ldots, 6$ defining an independent set that can be assigned with one color (orange), and the seven vertices $(0, i)$, for $i = 0, \ldots, 6$ defining another independent set that can be assigned with one color (green). The remaining edges of $C(1)$ can be assigned with the colors orange and green. Analogously for the Hamiltonian cycles $C(2), C(3), C(4), C(5)$ and $C(6)$, as in Figure **??**. The remaining uncolored vertices $(1, i)$, for $i = 0, 1, 2, 5$ and $(3, i)$, for $i = 3, 4, 6$ of Figure **??** represent an independent set that can be colored with the guiding color. Thus we can easily obtain a 13-total coloring of $C_5 \times K_7$ from the elements colored with the guiding color.

Elements of $C_5 \times K_7$ of the guiding color	
$C(1)$	$(4,3)(0,5), (4,2)(0,6), (4,0)(0,1), (4,5)(0,4)$ $(4,6)(0,3), (4,1)(0,2), (4,4)(0,0)$
$C(2)$	$(2,3)(3,1), (2,0)(3,2), (2,6)(3,5)$
$C(3)$	$(1,3)(2,4)$
$C(4)$	$(1,4)(2,5)$
$C(5)$	$(2,2)(3,0)$
$C(6)$	$(1,6)(2,1)$
Vertices: $(1,i)$, for $i = 0,1,2,5$ and $(3,i)$, for $i = 3,4,6$	

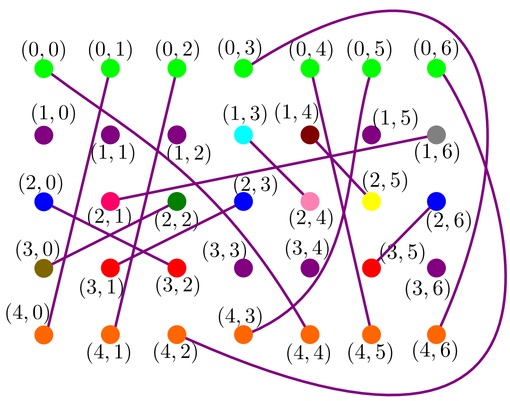

Figure 3: A table composed by the elements of the guiding purple color in $C_5 \times K_7$, and its depiction using colors of the endpoints to identify the Hamiltonian cycles containing them. Source: Authors.

3 Graphs $C_5 \times K_n$ Are Type 1, for Odd n and $gcd(5, n) = 1$

In this section, we present the main result of this paper. We apply the guiding color technique to find optimal total colorings of all $C_5 \times K_n$, for $n \geq 7$ when $gcd(m, n) = 1$. Furthermore, we present the 4 particular cases that are too small to satisfy the described pattern.

Theorem 3.1. *For odd number $n \geq 7$ and $gcd(5, n) = 1$, the graph $C_5 \times K_n$ is Type 1.*

Proof. To obtain a $(2n-1)$-total coloring for $C_5 \times K_n$ for an odd number $n \geq 7$ with $\gcd(5, n) = 1$, first we use the Hamiltonian decomposition of $C_5 \times K_n$ defined in Subsection 2.1 to construct the table respectively with the elements of the guiding color.

The general case for $C_5 \times K_n$ is obtained for $n \geq 7$ and $gcd(5, n) = 1$, as presented in Table 1. This case has 4 particular graphs: $C_5 \times K_7$ (presented in the table in Figure 3), $C_5 \times K_9$ (presented in Table 2), $C_5 \times K_{11}$ (presented in Table 3) and $C_5 \times K_{13}$ (presented in Table 4).

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Table 1: Elements of the guiding color $C_5 \times K_n$, for $n \geq 7$ and $gcd(5, n) = 1$.

Cycle	Edges	Cycle	Edges
$C(1)$	$(4, \sigma^0(C_n)_{(5t+4) \text{ mod } n})(0, \sigma^0(C_n)_{(5t+5) \text{ mod } n})$, $t = 0, \dots, n-1$	$C(7)$	$(2, 2)(3, 13)$
$C(2)$	$(2, 3)(3, 1), (2, n-2)(3, 6), (2, 8)(3, n-5)$	$C(n-3)$	$(2, 0)(3, n-3)$
$C(i)$	$(1, i)(2, i+1)$, $i = 3, \dots, n-2$ $i \neq 1, 2, 7, n-3, n-1$	$C(n-1)$	$(1, n-1)(2, 1)$
Vertices: $(1, i)$, for $i = 0, 1, 2, 7, n-3$ and $(3, i)$, for $i = 0, \dots, n-1$, $i \neq 1, 6, 13, n-5, n-3$			

Table 2: Elements of the guiding color in the particular graph $C_5 \times K_9$.

Cycle	Edges	Cycle	Edges
$C(1)$	$(4, 3)(0, 7), (4, 0)(0, 1), (4, 7)(0, 4), (4, 1)(0, 2), (4, 4)(0, 6),$ $(4, 2)(0, 8), (4, 6)(0, 5), (4, 8)(0, 3), (4, 5)(0, 0)$	$C(5)$	$(1, 5)(2, 6)$
$C(2)$	$(2, 3)(3, 1), (2, 7)(3, 6), (2, 1)(3, 4)$	$C(6)$	$(2, 2)(3, 0)$
$C(3)$	$(1, 3)(2, 4)$	$C(7)$	$(1, 7)(2, 8)$
$C(4)$	$(1, 4)(2, 5)$	$C(8)$	$(2, 0)(3, 8)$
Vertices: $(1, i)$, for $i = 0, 1, 2, 6, 8$ and $(3, i)$, for $i = 2, 3, 5, 7$			

Table 3: Elements of the guiding color in the particular graph $C_5 \times K_{11}$.

Cycle	Edges	Cycle	Edges
$C(1)$	$(4, 3)(0, 9), (4, 7)(0, 6), (4, 10)(0, 3), (4, 5)(0, 7), (4, 2)(0, 10), (4, 8)(0, 5),$ $(4, 1)(0, 2), (4, 4)(0, 8), (4, 0)(0, 1), (4, 9)(0, 4), (4, 6)(0, 0)$	$C(6)$	$(1, 6)(2, 7)$
$C(2)$	$(2, 3)(3, 1), (2, 9)(3, 6), (2, 2)(3, 3)$	$C(7)$	$(1, 7)(2, 8)$
$C(3)$	$(1, 3)(2, 4)$	$C(8)$	$(2, 0)(3, 8)$
$C(4)$	$(1, 4)(2, 5)$	$C(9)$	$(1, 9)(2, 10)$
$C(5)$	$(1, 5)(2, 6)$	$C(10)$	$(1, 10)(2, 1)$
Vertices: $(1, i)$, for $i = 0, 1, 2, 8$ and $(3, i)$, for $i = 0, 2, 4, 5, 7, 9, 10$			

Table 4: Elements of the guiding color in the particular graph $C_5 \times K_{13}$.

Cycle	Edges	Cycle	Edges
$C(1)$	$(4, 3)(0, 11), (4, 9)(0, 6), (4, 1)(0, 2), (4, 4)(0, 10), (4, 8)(0, 7), (4, 12)(0, 3), (4, 5)(0, 9),$ $(4, 0)(0, 1), (4, 11)(0, 4), (4, 6)(0, 8), (4, 2)(0, 12), (4, 10)(0, 5), (4, 7)(0, 0)$	$C(7)$	$(2, 0)(3, 7)$
$C(2)$	$(2, 3)(3, 1), (2, 11)(3, 6), (2, 8)(3, 0)$	$C(8)$	$(1, 8)(2, 9)$
$C(3)$	$(1, 3)(2, 4)$	$C(9)$	$(1, 9)(2, 10)$
$C(4)$	$(1, 4)(2, 5)$	$C(10)$	$(1, 7)(2, 2)$
$C(5)$	$(1, 5)(2, 6)$	$C(11)$	$(1, 11)(2, 12)$
$C(6)$	$(1, 6)(2, 7)$	$C(12)$	$(1, 12)(2, 1)$
Vertices: $(1, i)$, for $i = 0, 1, 2, 10$ and $(3, i)$, for $i = 2, 3, 4, 5, 8, 9, 10, 11, 12$			

Thus, the family $C_5 \times K_n$, with an odd number n and $gcd(5, n) = 1$, is Type 1. □

4 Conclusion

In this paper, we determine optimal total colorings for all $C_5 \times K_n$, establishing that all $C_5 \times K_n$ are Type 1, when n is odd and not a multiple of 5. This result uses the guiding color technique, which can be applied to all graphs having Hamiltonian decomposition.

Based on the evidence found so far, we conjecture that $C_m \times K_n$ is Type 2 if and only if m is not a multiple of 3 and $n = 2$.

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