Proceeding Series of the Brazilian Society of Computational and Applied Mathematics, v. 12, n. 1, 2026.

Trabalho apresentado no XLIV CNMAC, Fundação Getulio Vargas - Rio de Janeiro - RJ, 2025.

Proceeding Series of the Brazilian Society of Computational and Applied Mathematics

Numerical-Analytical Evidence of Convergence of the Semi-Discrete Lagrangian-Eulerian Numerical Method for the Korteweg-de Vries Equation

Eduardo Abreu[1]
IMECC/UNICAMP, Campinas, SP
Wanderson Lambert[2]
ICT/UNIFAL, Poços de Caldas, MG
Erivaldo Lima[3]
DMAT/UFRR, Boa Vista, RR; PPGMA/IMECC/UNICAMP, Campinas, SP

Abstract. In this work, we present a new extended formulation of the semi-discrete Lagrangian-Eulerian numerical method applied to the Korteweg-de Vries equation, which has a smooth convex flux function. This new scheme is applied to one-dimensional scalar problems incorporating a linear dispersive term with a constant dispersive coefficient. We present here some numerical experiments that provide strong evidence of the method's convergence. Whenever possible, a comparison is made between the numerical results and exact solutions or highly accurate approximations.

Keywords: Extended Lagrangian-Eulerian Approach; Semi-Discrete Method; Korteweg-de Vries Equation; Dispersive Conservation Law

1 Introduction

Consider the initial value problem

$$\begin{cases} \partial_t u + \partial_x(H(u)) = \mu \partial_x^3 u, & (x,t) \in \mathbb{R} \times (0,T], \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1}$$

where u_0 is the initial condition, $\mu \neq 0$ is the dispersive constant and the third derivative is bounded. Equation (1) represents the **Korteweg-de Vries** (KdV) equation when the convex flux function is $H(u) = \varepsilon u^2$ with ε non-zero. The KdV equation, introduced in 1895 by Korteweg and de Vries [9], is widely applied to describe wave phenomena, such as shallow water waves [9], and bubble-liquid mixtures [14], among others. Analytical solutions have been developed for specific problems [6], but numerical methods are critical for broader applications. Since analytical solutions are feasible only for limited initial conditions, numerical approaches play a crucial role in studying the physical phenomena governed by these equations.

Several recent works [2–4] have introduced and expanded Lagrangian-Eulerian numerical methods, in both the fully discrete (FDLE) and semi-discrete (SDLE) versions, for solving hyperbolic conservation and balance law problems. In this work, we develop a semi-discrete Lagrangian-Eulerian method for equation (1), where the flux function H characterizes the KdV equation as described above.

[1]eabreu@unicamp.br
[2]wanderson.lambert@unifal-mg.edu.br
[3]erivaldo.lima@ufrr.br

The objective of this study is to provide numerical evidence of convergence for the semi-discrete Lagrangian-Eulerian method in (1), focusing on problems that have been well studied in the literature [5, 7, 15]. Additionally, we investigate the order of convergence of the proposed method in several examples using the L^2 and L^∞ norms.

2 Semi-Discrete Lagrangian-Eulerian Numerical Scheme for Korteweg-de Vries Equations

Due to the relevance of this type of equation, we propose here our formulation *Semi-Discrete Lagrangian-Eulerian (SDLE) for Korteweg-de Vries equations*

$$\frac{d}{dt}u_j(t) = -\frac{1}{\Delta x}\left[\mathcal{F}(u_j, u_{j+1}) - \mathcal{F}(u_{j-1}, u_j)\right] + \mu(u_{xxx})_j, \tag{2}$$

where numeric flux $\mathcal{F}(u_j, u_{j+1})$ is given by

$$\mathcal{F}(u_j, u_{j+1}) = \frac{1}{4}\left[b_{j+\frac{1}{2}}\left(u^-_{j+\frac{1}{2}} - u^+_{j+\frac{1}{2}}\right) + (f_j + f_{j+1})\left(u^-_{j+\frac{1}{2}} + u^+_{j+\frac{1}{2}}\right)\right], \tag{3}$$

with the slope of the *no-flow-curve* $f_j = \frac{H(u_j)}{u_j}$ and

$$u^-_{j+\frac{1}{2}} = u_j + \frac{\Delta x}{4}\left(u_x\right)_j \quad \text{and} \quad u^+_{j+\frac{1}{2}} = u_{j+1} - \frac{\Delta x}{4}\left(u_x\right)_{j+1}. \tag{4}$$

Here, $\Delta x > 0$ represents the spatial grid spacing, while $\left(u_x\right)_j$ and $\left(u_{xxx}\right)_j$ are approximations of the first and third derivatives, respectively, and $b_{j+\frac{1}{2}} = b_{j+\frac{1}{2}}\left(f_j, f_{j+1}\right)$ is a function of order $\mathcal{O}(\frac{H(u)}{u})$.

The construction of this numerical method is carried out in a general form in the thesis work [10], which is still in progress. The semi-discrete Lagrangian-Eulerian method (2)-(4) extends the purely hyperbolic case presented in [2]. For the construction of the no-flow curve, we reference [1].

For convenience, we employ linear reconstruction [11]. Additionally, high-order reconstructions of the numerical flux, as proposed in [3], can yield solutions with high-order convergence and improved resolution.

For the preliminary CFL stability condition, we define the function $F(u) = H(u) - \mu\frac{\partial^2 u}{\partial x^2}$. Thus, we can rewrite equation (1) in its conservative form as $\frac{\partial u}{\partial t} + \frac{\partial F(u)}{\partial x} = 0$. The authors in [2] demonstrated the CFL stability condition for the Lagrangian-Eulerian numerical method with linear reconstruction applied to the hyperbolic problem. Based on this condition, we derive the CFL stability condition for equation (1).

$$\frac{\Delta t^n}{\Delta x}\left(\max_j\left|\frac{H(u_j)}{u_j}\right| + \frac{4|\mu|}{\Delta x^2}\right) \leqslant \frac{1}{2}. \tag{5}$$

3 Numerical-Analytical Evidence of Convergence

For our numerical experiments, we employ centered finite difference approximations for the first and third derivatives, i.e.,

$$(u_x)_j = \frac{1}{2\Delta x}\left(u_{j+1} - u_{j-1}\right) \text{ and } (u_{xxx})_j = \frac{1}{2\Delta x^3}\left(-u_{j-2} + 2u_{j-1} - 2u_{j+1} + u_{j+2}\right). \tag{6}$$

We use the following expression for $b_{j+\frac{1}{2}}$ in the numerical flux function (3)

$$b_{j+\frac{1}{2}} = \max_j\left\{|\zeta_1 f_j + \zeta_2 f_{j+1}|\right\}, \quad \zeta_1, \zeta_2 \geqslant 0, \tag{7}$$

subject to the global CFL stability condition (5). Here, we are also interested in achieving high-order temporal accuracy. For this reason, we reformulate the *Runge-Kutta-Shu* method [13] to obtain the *Runge-Kutta-Shu-Leap-Frog* method for dispersive problems

$$
\begin{aligned}
u_j^* &= u_j^{n-1} - \frac{2\Delta t^n}{\Delta x}\left[\mathcal{F}(u_j^n, u_{j+1}^n) - \mathcal{F}(u_{j-1}^n, u_j^n)\right] + 2\Delta t^n \mu (u_{xxx}^n)_j,\\
u_j^{**} &= u_j^{n} - \frac{2\Delta t^n}{\Delta x}\left[\mathcal{F}(u_j^*, u_{j+1}^*) - \mathcal{F}(u_{j-1}^*, u_j^*)\right] + 2\Delta t^n \mu (u_{xxx}^*)_j,\\
u_j^{n+1} &= \frac{1}{2}\left[u_j^n + u_j^{**}\right].
\end{aligned} \tag{8}
$$

For $p = 2, \infty$, the definitions of the relative errors $\|e\|_{L^p}$ (between consecutive meshes) can be found in [7], while the definitions of the estimated order of convergence EOC_p^* can be found in [2].

Example 1. *The Korteweg-de Vries equation, in the form (1) with* $H(u) = \frac{u^2}{2}$, *is written as:*

$$
\begin{cases}
\partial u_t + \partial(u^2/2)_x = \mu \partial_{xxx} u,\\
u(x,0) = 3c\,\mathrm{sech}^2\left(0.5\sqrt{-\frac{c}{\mu}}x - 6\right), \quad x \in [0,2],
\end{cases} \tag{9}
$$

where $\mu = -4.84 \times 10^{-4}$ *and* $c = 0.3$. *This problem, along with its exact solution* $u(x,t) = 3c\,\mathrm{sech}^2\left(0.5\sqrt{-\frac{c}{\mu}}(x-ct) - 6\right)$, *can be found [7].*

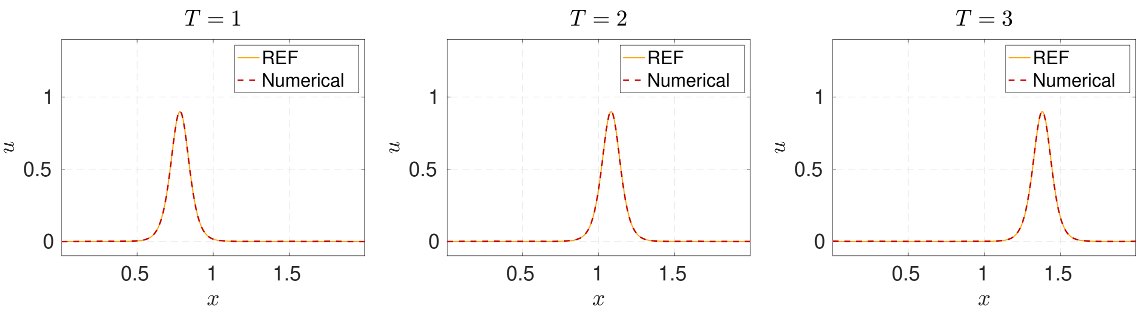

Figure 1: Example 1 using the SDLE, with mesh $m = 256$, $\zeta_1 = \zeta_2 = 0$, $CFL = 0.3$. Here, the reference (REF) is given by the exact solution.

Table 1: Performance at solving the Example 1. We have $T = 3.0$ with successive meshes, $\zeta_1 = \zeta_2 = 0$ in (7), $CFL = 0.3$, relative errors $\|e\|_{L^p}$ and estimated order of convergence EOC_p^* ($p = 2$ or $p = \infty$) in the L^2 and L^∞ spaces.

T	m	Δx	$\|e\|_{L^2}$	EOC_2^*	$\|e\|_{L^\infty}$	EOC_∞^*
	256	7.812500e-03	7.493282e-04	1.8045	2.326000e-02	1.3319
3.0	512	3.906250e-03	2.145200e-04	1.4966	9.240000e-03	1.0348
	1024	1.953125e-03	7.602461e-05	1.2076	4.510000e-03	0.8002

Example 2. *The Korteweg-de Vries equation in the form (1) with* $H(u) = \frac{u^2}{2}$ *is written as:*

$$
\begin{cases}
\partial u_t + \partial(u^2/2)_x = \mu \partial_{xxx} u,\\
u(x,0) = -12\mu \dfrac{\partial^2}{\partial x^2}(\log(F)), \quad x \in [0,4],
\end{cases} \tag{10}
$$

DOI: 10.5540/03.2026.012.01.0318

where $F = 1+e^{\nu_1}+e^{\nu_2}+\eta^2 e^{\nu_1+\nu_2}$, $\nu_i = a_i x+\nu_i$, $i \in \{1,2\}$, $\eta = \left(\frac{a_1-a_2}{a_1+a_2}\right)$, $a_1 = \sqrt{-\frac{c_1}{\mu}}$, $a_2 = \sqrt{-\frac{c_2}{\mu}}$, *with* $\mu = -4.84 \times 10^{-4}$, $c_1 = 0.3$, $c_2 = 0.1$, $\nu_1 = -0.48a_1$ *and* $\nu_2 = -1.07a_2$. *The exact solution is obtained for* $\nu_i = a_i x - a_i^3 \mu t + \nu_i$, *with* $i \in \{1,2\}$. *This problem, along with its exact solution, can be found in [5].*

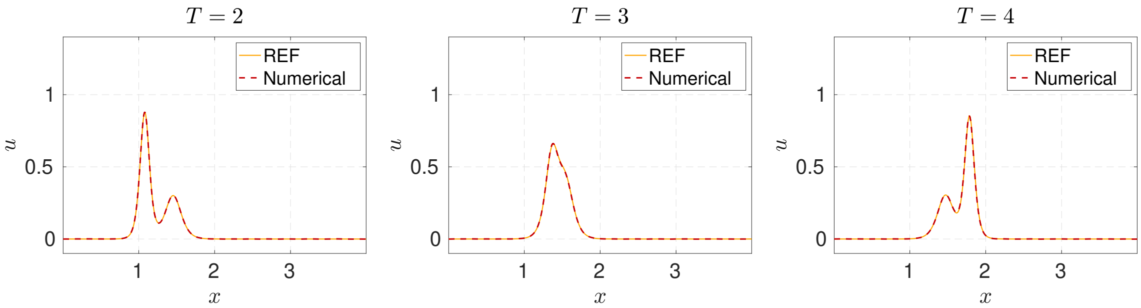

Figure 2: Example 2 using the SDLE, with mesh $m = 512$, $\zeta_1 = \zeta_2 = 0$ and $CFL = 0.3$. Here, the reference (REF) is given by the exact solution.

Table 2: Performance in solving Example 2. We consider $T = 4.0$ with successive meshes, $\zeta_1 = \zeta_2 = 0$ in (7), and $CFL = 0.3$, relative errors $\|e\|_{L^p}$ and estimated order of convergence EOC_p^* ($p = 2$ or $p = \infty$) in the L^2 and L^∞ spaces.

T	m	Δx	$\|e\|_{L^2}$	EOC_2^*	$\|e\|_{L^\infty}$	EOC_∞^*
	512	7.812500e-03	9.178823e-04	1.9744	2.919000e-02	1.5056
4.0	1024	3.906250e-03	2.335852e-04	1.6359	1.028000e-02	1.2145
	2048	1.953125e-03	7.516097e-05	1.3186	4.430000e-03	1.0164

Example 3. *The Korteweg-de Vries equation in the form (1) with* $H(u) = \frac{u^2}{2}$ *is written as:*

$$\begin{cases} \partial u_t + \partial(u^2/2)_x = \mu \partial_{xxx} u, \\ u(x,0) = \cos(\pi x), \quad x \in [0,2], \end{cases} \tag{11}$$

where $\mu = -4.84 \times 10^{-4}$. *This problem can be found in [15].*

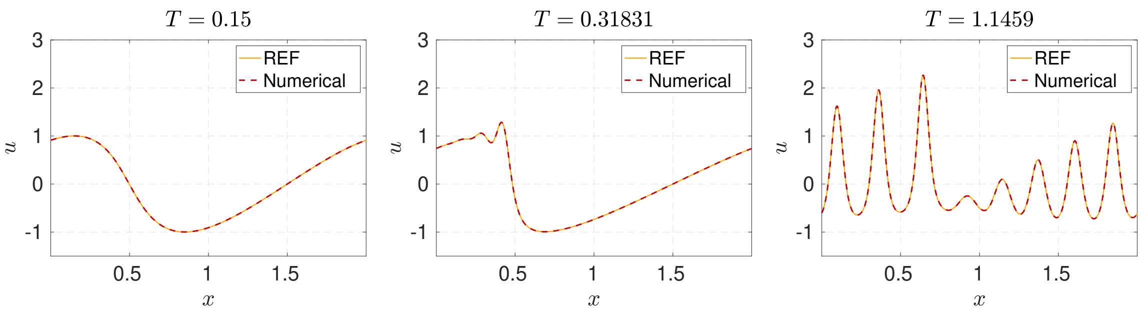

Figure 3: Example 3 using the SDLE, with mesh $m = 1024$, $\zeta_1 = \zeta_2 = 0$ and $CFL = 0.3$. Here, the reference (REF) is the numerical solution of the method proposed by [15] on a very fine mesh.

DOI: 10.5540/03.2026.012.01.0318

Table 3: Performance in solving Example 3. We consider $T = 1.11459$ $(3.6/\pi)$ with successive meshes, $\zeta_1 = \zeta_2 = 0$ in (7), and $CFL = 0.3$, relative errors $\|e\|_{L^p}$ and estimated order of convergence EOC_p^* ($p = 2$ or $p = \infty$) in the L^2 and L^∞ spaces.

T	m	Δx	$\|e\|_{L^2}$	EOC_2^*	$\|e\|_{L^\infty}$	EOC_∞^*
	256	7.812500e-03	6.340419e-03	2.5175	1.228625e-01	1.5659
$3.6/\pi$	512	3.906250e-03	1.107341e-03	1.5024	4.150000e-02	0.9690
	1024	1.953125e-03	3.908468e-04	1.2717	2.120000e-02	0.9597

Example 4. *The Korteweg-de Vries equation in the form (1) with* $H(u) = \frac{u^2}{2}$ *is written as:*

$$\begin{cases} \partial u_t + \partial(u^2/2)_x = \mu \partial_{xxx} u, \\ u(x,0) = 12 \sum_{i=1}^{5} c_i^2 \operatorname{sech}^2\left(c_i(x - x_i)\right), \quad x \in [-150, 150], \end{cases} \tag{12}$$

where $\mu = -1.0$, $c_1 = 0.3$, $c_2 = 0.25$, $c_3 = 0.2$, $c_4 = 0.15$, $c_5 = 0.1$, $x_1 = -120$, $x_2 = -90$, $x_3 = -60$, $x_4 = -30$ *and* $x_5 = 0$. *This problem can be found in [5].*

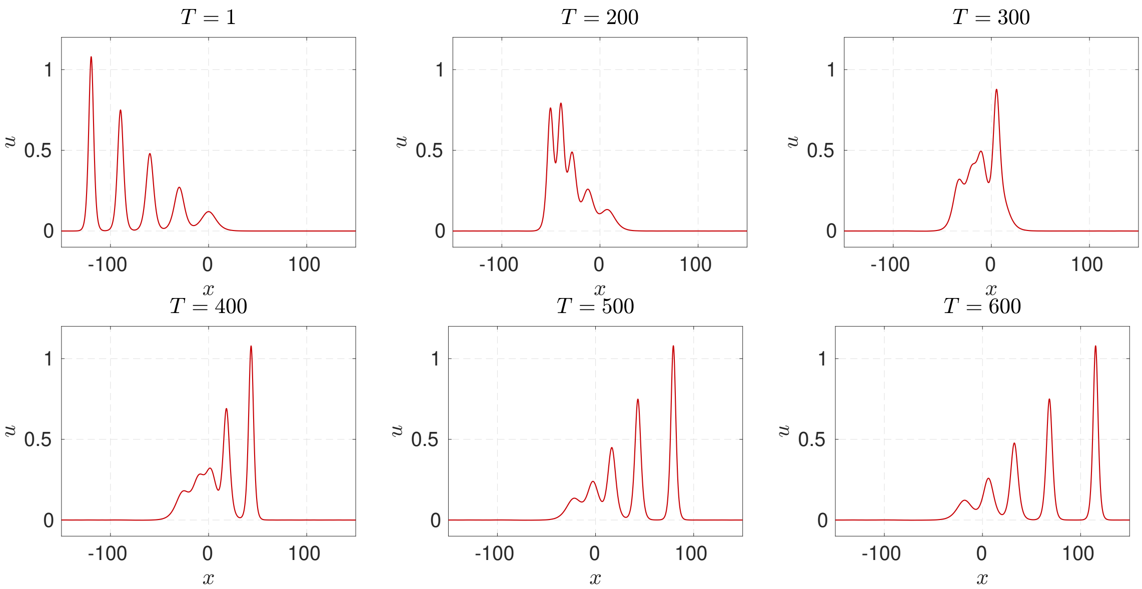

Figure 4: Example 4 using the SDLE method, with $\zeta_1 = \zeta_2 = 0$ in (7) and $CFL = 0.3$. Here, $\Delta x = 1.464844 \times 10^{-1}$ ($m = 2048$), and Δt^n is subject to the CFL stability condition (5).

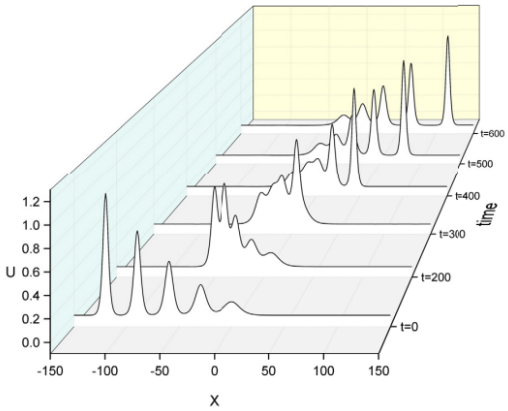

Figure 5: References for the solution of Example 4. Source: Article [5].

4 Final Remarks

In this work, we have verified the simplicity, versatility, robustness, and potential of the extended Semi-Discrete Lagrangian-Eulerian numerical method for the KdV equation. The numerical solutions obtained in the experiments demonstrate high consistency. We present and discuss numerical experiments involving complex problems, providing strong numerical evidence that validates the convergence of the method.

Our approach has also been successfully applied to the modified Korteweg-de Vries (mKdV) equations and to two-dimensional dispersive conservation law equations. As a next step, we aim to generalize our approach to other intricate diffusive-dispersive models in two- and three-dimensional spaces to address state-of-the-art questions in nonlinear modeling related to internal waves (e.g., the Benjamin-Ono and Intermediate Long Wave equations, as in [12]) and to study soliton stability and interactions in 3D (e.g., the Zakharov-Kuznetsov equation, as in [8]).

This work is part of the academic thesis study of the PhD student Erivaldo Diniz de Lima (2022–in progress, expected defense in February 2026). The thesis topic at PPGMA/IMECC (Applied Mathematics) is: *A Semi-Discrete Lagrangian-Eulerian Method for Diffusive-Dispersive Conservation Laws.*

Acknowledgments

Eduardo Abreu acknowledges the financial support received from the National Council for Scientific and Technological Development (CNPq) (Grant Nº. 307641/2023-6).

Erivaldo Lima acknowledges the Federal University of Roraima (UFRR) for the leave granted to pursue his doctoral studies.

References

[1] E. Abreu, J. François, W. Lambert, and J. Pérez. "A class of positive semi-discrete Lagrangian–Eulerian schemes for multidimensional systems of hyperbolic conservation laws". In: **Journal of Scientific Computing** 90.1 (2022), p. 40.

[2] E. Abreu, J. François, W. Lambert, and J. Pérez. "A semi-discrete Lagrangian–Eulerian scheme for hyperbolic-transport models". In: **Journal of Computational and Applied Mathematics** 406 (2022), p. 114011.

[3] E. Abreu, E. Pandini, and W. Lambert. “An Enhanced Lagrangian-Eulerian Method for a Class of Balance Laws: Numerical Analysis via a Weak Asymptotic Method With Applications”. In: **Numerical Methods for Partial Differential Equations** 41.1 (2025), e23163.

[4] E. Abreu and J. Pérez. “A fast, robust, and simple Lagrangian–Eulerian solver for balance laws and applications”. In: **Computers & Mathematics with Applications** 77.9 (2019), pp. 2310–2336.

[5] A. Başhan. “Modification of quintic B-spline differential quadrature method to nonlinear Korteweg-de Vries equation and numerical experiments”. In: **Applied Numerical Mathematics** 167 (2021), pp. 356–374.

[6] B. Fornberg and G. B. Whitham. “A numerical and theoretical study of certain nonlinear wave phenomena”. In: **Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences** 289.1361 (1978), pp. 373–404.

[7] S-Y. Hao, S-S. Xie, and S-C. Yi. “The Galerkin method for the KdV equation using a new basis of smooth piecewise cubic polynomials”. In: **Applied Mathematics and Computation** 218.17 (2012), pp. 8659–8671.

[8] C. Klein, S. Roudenko, and N. Stoilov. “Numerical study of soliton stability, resolution and interactions in the 3D Zakharov–Kuznetsov equation”. In: **Physica D: Nonlinear Phenomena** 423 (2021), p. 132913.

[9] D. J. Korteweg and G. D. Vries. “XLI. On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves”. In: **The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science** 39.240 (1895), pp. 422–443.

[10] E. Lima. “A Semi-Discrete Lagrangian-Eulerian Metohd for Diffusive-Dispersive Conservation Laws”. Doctoral thesis. UNICAMP, expected date defense to February/2026.

[11] H. Nessyahu and E. Tadmor. “Non-oscillatory central differencing for hyperbolic conservation laws”. In: **Journal of Computational Physics** 87.2 (1990), pp. 408–463. ISSN: 0021-9991.

[12] J-C. Saut. “Benjamin-Ono and intermediate long wave equations: Modeling, IST and PDE”. In: **Nonlinear dispersive partial differential equations and inverse scattering** (2019), pp. 95–160.

[13] C. W. Shu and S. Osher. “Efficient implementation of essentially non-oscillatory shock-capturing schemes”. In: **Journal of Computational Physics** 83 (1989), pp. 32–78.

[14] L. V. Wijngaarden. “On the equations of motion for mixtures of liquid and gas bubbles”. In: **Journal of fluid mechanics** 33.3 (1968), pp. 465–474.

[15] N. J. Zabusky and M. D. Kruskal. “Interaction of "solitons" in a collisionless plasma and the recurrence of initial states”. In: **Physical review letters** 15.6 (1965), p. 240.

DOI: 10.5540/03.2026.012.01.0318