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Optimal Total Coloring a New Infinite Family of Snarks Obtained with Kochol's Superposition

Rieli Araujo[1], Celina M.H. de Figueiredo [2]
PESC/UFRJ, Rio de Janeiro, RJ
Diana Sasaki[3]
IME/UERJ, Rio de Janeiro, RJ
Simone Dantas[4]
IME/UFF, Niterói, RJ

Abstract. Snarks are cubic graphs with peculiar properties, making it very difficult to construct new snarks. In 1996, Kochol introduced a method that allows us to obtain a new snark from two known snarks; this construction is called Kochol's superposition. We were able to construct an infinite family of girth 4 snark through Kochol's superposition of Flower snarks (known to be Type 1) and a girth 4 snark (known to be Type 2), which allowed us to obtain new snarks from known ones. This work shows that all members of this new family are Type 2.

Keywords. Snarks, Kochol's Superposition, Total Coloring

1 Introduction

Snarks are a special class of cubic graphs, whose first known example is the Petersen graph during the Four Color Conjecture in 1880 by Tait [13]. In nearly one hundred years of search, only five snarks were identified. In 1976, Martin Gardner [6], inspired by the difficulty of finding these graphs, named them snarks, drawing inspiration from "The Hunting of the Snark" [5].

Over time, several definitions have been proposed for snarks. In this paper, we adopt the definition in which snark is a cubic, cyclically 4-edge-connected, and Class 2 graph, i.e., that cannot be 3-edge-colored. A graph is cyclically 4-edge-connected when the removal of at least 4 edges is necessary to disconnect it, and the resulting connected components must contain cycles. For further details on snark properties, we refer to [2].

In 1975, Isaacs [7] introduced an operation known as the **dot product**, which allowed for the construction of new snarks from known snarks and the creation of the first infinite families of snarks. In 1996, Kochol [10] proposed a new method for constructing new snarks from smaller graphs, called **Kochol's superposition**, typically used to obtain snarks with large girth. Many authors restrict the definition of snarks to graphs with girth at least 5, allowing girth 4 can lead to the trivial construction of infinitely many cubic graphs. Brinkmann et al. [3] showed that it is possible to construct a snark with girth 4. Girth has also proven to be relevant in other coloring contexts, such as fractional total colorings [9]. Our goal is to apply Kochol's superposition, which is commonly used to construct snarks with large girth (at least 5), to generate new snarks with girth 4. To this end, we apply it to a well-known infinite family of Flower snarks and the smallest Type 2 snark of girth 4, to obtain a new infinite family of Type 2 snarks with small girth.

[1]rieli@cos.ufrj.br
[2]celina@cos.ufrj.br
[3]diana.sasaki@ime.uerj.br
[4]sdantas@id.uff.br

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2 Preliminaries

A **semigraph** is a 3-tuple $G = (V, E, S)$, where $V(G)$ is a set of vertices of G, $E(G)$ is a set of edges disjoint from $V(G)$, and $S(G)$ is a set of **semiedges** having only one endpoint belonging to $V(G)$. When it is clear, we simply denote $V(G)$, $E(G)$ and $S(G)$ by V, E and S, respectively. An edge $e \in E$ whose endpoints are vertices v and w is denoted by vw.

Graph coloring involves the partition of its elements (edges and/or vertices) into a family of disjoint sets $\mathcal{C} = \{c_1, c_2, \ldots, c_k\}$, called **color classes**, where k is the number of colors assigned to a graph G and c_i for $i \in \{1, 2, ...k\}$ is a **color class**. A **total coloring** is an assignment of colors to both its vertices and its edges such that no adjacent edges, adjacent vertices, incident vertices or edges share the same color. When k colors are assigned to the elements of G, we say that G has a ***k*-total coloring**. The **total chromatic number** of G is the smallest value of k necessary to obtain a total coloring, denoted by $\chi''(G)$.

Clearly, $\chi''(G) \geq \Delta(G)+1$, where $\Delta(G)$ is the maximum degree of G. Independently, Behzad [1] and Vizing [15] conjectured an upper bound for the total chromatic number.

Conjecture 2.1 (Total Coloring Conjecture (TCC) [1, 15])**.** *For every simple graph G, the total chromatic number of G satisfies*

$$\chi''(G) \leq \Delta(G) + 2.$$

Since $\chi''(G) \geq \Delta(G) + 1$, and according to the Total Coloring Conjecture (TCC), $\chi''(G) \leq \Delta(G) + 2$, it is conventionally established that: when $\chi''(G) = \Delta(G) + 1$, the graph is called **Type 1**; and when $\chi''(G) = \Delta(G) + 2$, the graph is called **Type 2**. Determining whether a graph is Type 1 or Type 2 is an NP-hard problem [12]. Although the conjecture is still open, its validity has been confirmed for some classes of graphs. Independently, Rosenfeld [11] and Vijayaditya [14] demonstrated its validity for cubic graphs. In this case, it can be stated that if G is a cubic graph, its total chromatic number will be either 4 (Type 1) or 5 (Type 2).

3 Kochol's Superposition

In Kochol's superposition, the set is partitioned into n non-empty and pairwise disjoint sets $S_1, S_2, \ldots, S_n$, such that $|S_i| = k_i$ for $i \in \{1, 2, \ldots, n\}$. The sets S_i are called **connectors** of the k_n-semigraph M. This method consists of two main elements: the **superedge** and the **supervertex**. A **superedge** is a semigraph with two connectors and a **supervertex** is a semigraph with three connectors.

A superedge can be obtained by removing two non-adjacent vertices v and u from a snark, generating a semigraph with two connectors, where each connector has three semiedges, as shown in Figure 1, or a superedge can be an isolated edge.

A supervertex is a structure that connects superedges to the cubic graph G to create a new cubic graph. It can be a semigraph J', consisting of two isolated semiedges and one vertex that is an endpoint of 3 semiedges, or a semigraph K', consisting of a vertex that is an endpoint of 3 semiedges. These structures are illustrated in Figure 2. Using his own method, Kochol constructed infinite families of Type 1 snarks with large girth. Our objective is to apply Kochol's method to construct new Type 2 snarks with small girth, by Kochol's superposition of a Type 1 snark with a Type 2 snark.

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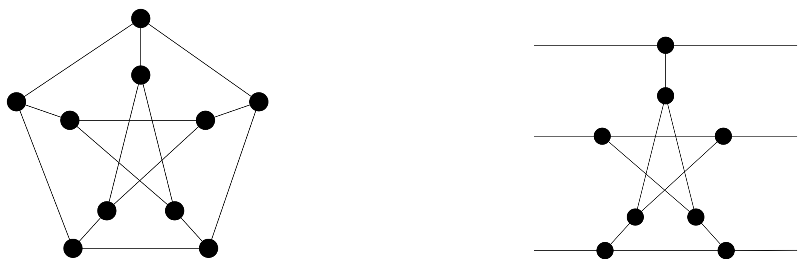

Figure 1: A superedge of the Petersen graph, obtained by removing two non-adjacent vertices. Source: created by the authors.

Figure 2: Supervertex J' and supervertex K'. Source: created by the authors.

4 Our Main Result

We consider Kochol's superposition of the semigraph obtained from Type 2 snark with girth 4 constructed by Brinkmann, Preissmann and Sasaki [3], which we refer to as the *brick's snark*, with the members of the well-known family of Flower snarks (proved to be Type 1 [4]).

Brick's snark was constructed by Brinkmann et al. [3] through a junction of the Type 2 cubic semigraph brick B^* with a Class 2 brick, resulting in a Type 2 snark, as illustrated in Figure 3a. The copies of the brick's snark are the superedges (ξ), and they were obtained by removing non-adjacent vertices v and u, illustrated in Figure 3b.

The brick B^*, illustrated in Figure 4, is the smallest Type 2 brick, consisting of 22 vertices, and it is unique up to isomorphism [3]. The resulting brick snark is also Type 2, as this property is inherited from the semigraph B^*.

Flower snarks The Flower snark family was the first infinite family introduced by Isaacs [8]. The first member of this family has 20 vertices and 30 edges and is denoted by J_5 because its center consists of a cycle of length 5, with each vertex in the center connected to a **"petal"**, as illustrated in Figure 5. The snark J_5, illustrated with one of its petals highlighted, is the first member of the Flower snark family J_i, defined for odd integers $i \geq 5$. Each snark J_i consists of i petals symmetrically attached to a central cycle of order i. The total number of vertices in J_i is $4i$.

We obtain a new infinite family of small girth snarks by applying Kochol's superposition to the superedges ξ with the Flower snarks. Kochol's superposition occurs as follows: the cycle formed by the vertices u_1, u_2, u_3, u_4 and u_5 (Figure 5) of each Flower snark form a supervertex, and each superedge ξ will be added between each pair of adjacent vertices. The graph obtained by joining the superedges and supervertices as in Figure 6 is called the **Flower brick**. We prove that all Flower bricks are Type 2, resulting in Theorem 4.1, presented below. The members of this family will be denoted by Fb_i, for $i = 5, 7, 9 \ldots$, thus the first member is Fb_5.

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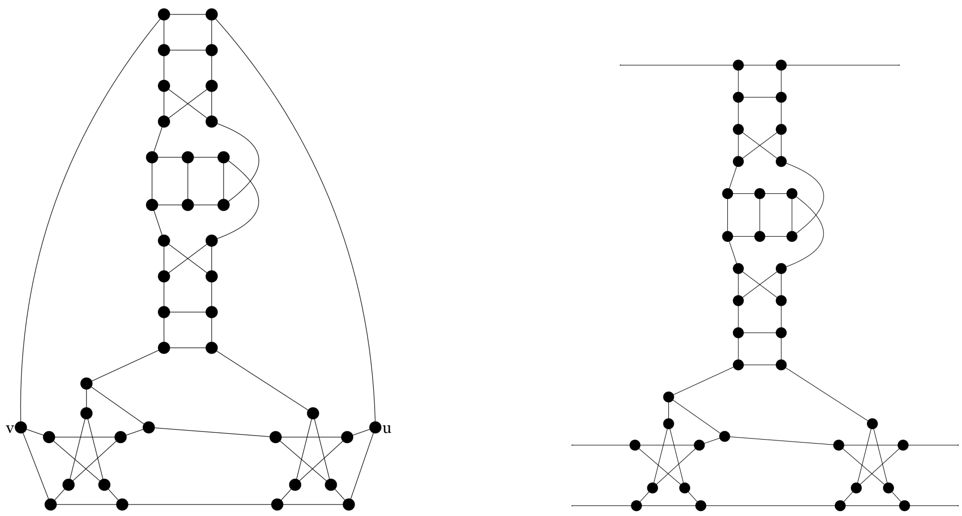

(a) Brick's snark. (b) Superedge ξ.

Figure 3: Girth 4 Type 2 snark and a superedge brick's snark. Source: created by the authors.

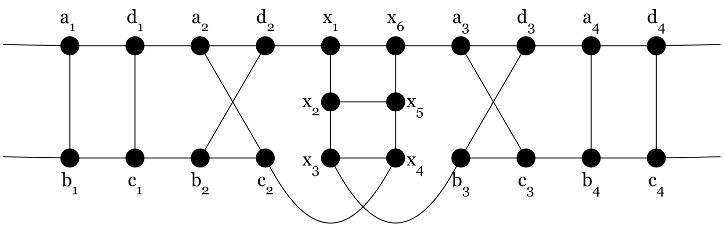

Figure 4: Brick semi-graph B^*. Source: created by the authors.

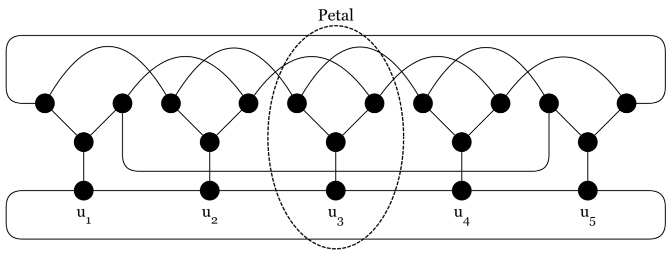

Figure 5: Flower snark J_5. Source: created by the authors.

Theorem 4.1. *All members of the Flower brick Fb_i snarks obtained by Kochol's superposition of the Flower snark J_i, odd $i \geq 5$ (Type 2) with a brick's snark (Type 1) are Type 2.*

Sketch of proof We refer to Figures 5 and 6. We select the cycle formed by the vertices u_1, u_2, u_3, u_4 and u_5. For each edge of this cycle, we include a copy of the superedge ξ. The number of superedges equals the number of petals in the snark (for example, J_5 has 5 petals, so 5 superedges are added). Furthermore, to achieve a Kochol's superposition, it is necessary to create

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a structure of supervertices. For this, we add two more edges.

First, we obtain a 4-total coloring for J_5 and for a pair of petals. This way, we can input the pair of colored petals between the second and third petals, thus maintaining a 4-total coloring for all members of this family. Each pair of petals added to the previous member maintains its total coloring, following the same pattern. An example of Kochol's superposition of the Flower snark J_5 with five copies of the Type 2 brick's snark is presented in Figure 6.

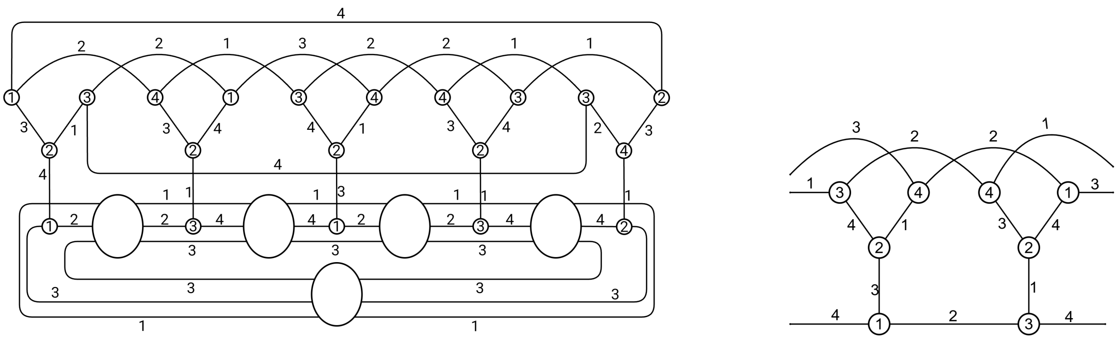

Figure 6: 4-total colorings of Flower brick Fb_5 and of a connected pair of petals. Source: created by the authors.

It is known that the superedge ξ does not admit a 4-total coloring. Therefore, when it is inserted through Kochol's superposition, the resulting graph is a Type 2 snark, as the semigraph B^* is preserved in the process, ensuring that the Type 2 property is maintained. Consequently, all members of the family obtained through Kochol's superposition are Type 2. To ensure that these colorings are preserved in all members of the family, we construct three distinct 5-total colorings of the superedge ξ. The colorings, denoted ξ_1, ξ_2, and ξ_3, are respectively illustrated in Figure 7. Thus, the coloring can be propagated recursively throughout all members of the family.

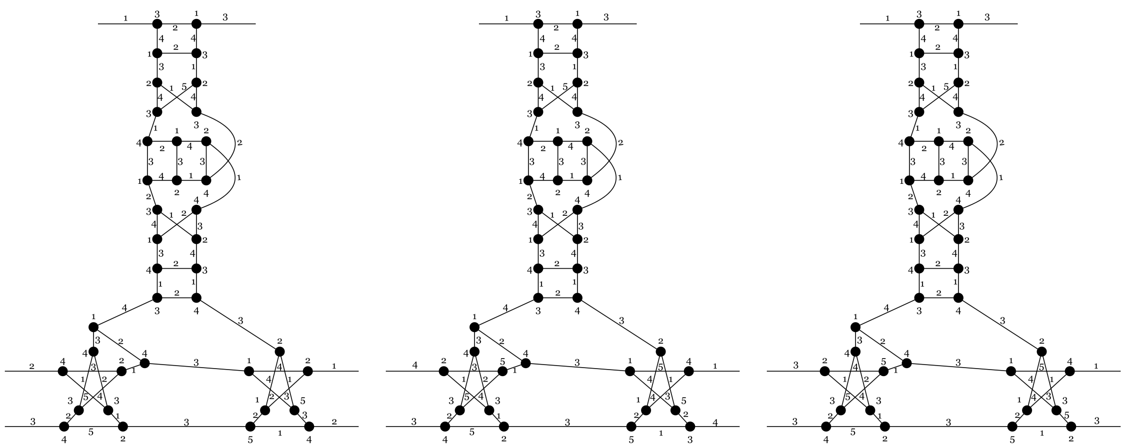

Figure 7: 5-total colorings of superedges ξ_1, ξ_2 and ξ_3. Source: created by the authors.

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5 Conclusion

We demonstrated that Kochol's superposition, typically applied to construct snarks with large girth, can also be adapted to generate an infinite family of Type 2 snarks with girth 4. Our aim is to further explore applications of Kochol's superposition beyond the traditional constraints, and to investigate whether Kochol's superposition between two Type 1 snarks can result in a Type 2 snark.

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