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Random Walks on Finite Discrete Velocity Fields

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Abstract. This work presents a discrete-time random velocity field on a class of one-dimensional finite lattices with periodic boundary conditions. We define the Eulerian and Lagrangian location processes, analyzing their relationship through circulant and permutation transition matrices. By examining the second-largest eigenvalue modulus, we characterize the convergence rate of the Lagrangian location process to its invariant distribution. We explore how spatial domain size and parity influence convergence behavior, providing insights into stochastic transport dynamics in discrete settings.

Keywords. Discrete Velocity Fields, Lagrangian Processes, Markov Chains, Eigenvalues.

1 Introduction

An important problem in statistical fluid mechanics is to obtain the statistical description of the motion of a single particle in a random velocity field. A fundamental example is the passive tracer transport problem, where the particle's motion, governed by

$$\frac{d\mathbf{X}_t}{dt} = \mathbf{U}(\mathbf{X}_t, t), \quad t > 0; \quad \mathbf{X}_0 = \mathbf{0}, \tag{1}$$

does not influence the velocity field $\mathbf{U}$. The main objective is to derive the probability law of the particle's position $\mathbf{X}_t$, which means the entire process $\{\mathbf{X}_t, t \geq 0\}$, from the law of $\mathbf{U}$ [2, 3, 6, 7].

Closely related to the passive tracer problem is the task of determining the law of the Lagrangian velocity process $\{\mathbf{U}(\mathbf{X}_t, t) : t \geq 0\}$, which describes the velocity experienced by an observer moving with the particle. An analogous discrete framework was introduced by Bennett and Zirbel [1] for a broad class of Eulerian velocity fields U on a periodic lattice in discrete time. Assuming that $\mathbf{U}$ is homogeneous and Markovian in time, it can be decomposed into the velocity type $\mathbf{I}$ and the location $\mathbf{L}$, such that $(\mathbf{I}, \mathbf{L})$ forms a Markov chain. Moreover, the generalized Lagrangian velocity V exhibits the same hidden Markov structure—with its location parameter $\mathbf{M}$ evolving differently from $\mathbf{L}$. Bennett and Zirbel [1] argued that the discrete space-time approach provides a meaningful simplification of the continuous model, with the transition matrix of $(\mathbf{I}, \mathbf{M})$ derived explicitly in terms of $(\mathbf{I}, \mathbf{L})$ and encoding information about the rate of convergence to equilibrium.

In Figure 1, we present discrete approximations of a two-dimensional continuous velocity field for various lattice sizes. Bennett and Zirbel [1] further argued that a discrete space-time approach can provide a meaningful simplification of the continuous model. For instance, the discrete equation of motion can be written as

$$\mathbf{X}_{t+1} = \mathbf{X}_t + \mathbf{U}_t(\mathbf{X}_t), \quad t = 0, 1, 2, \ldots; \quad \mathbf{X}_0 = \mathbf{0}, \tag{2}$$

which preserves the nonlinear relationship between $\mathbf{U}$ and $\mathbf{X}_t$ as in the continuous case.

In this work, we focus our attention on a subclass of one-dimensional velocity fields defined on a finite lattice with periodic boundary conditions. We also establish the type and location processes, as well as the transition matrices for $\mathbf{L}$ and $\mathbf{M}$, which allow us to discuss and analyze the convergence rate of the Lagrangian location process $\mathbf{M}$.

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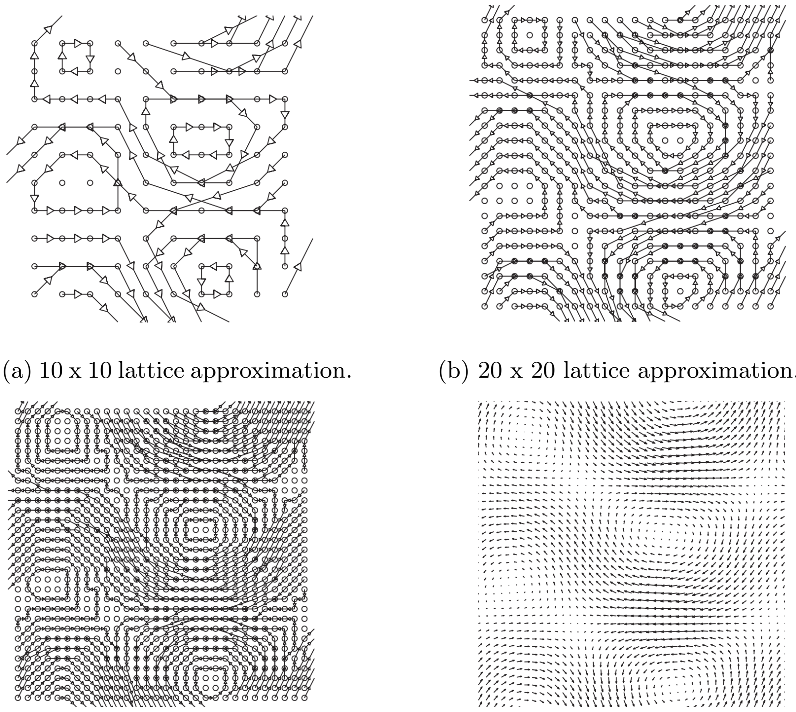

(a) 10 x 10 lattice approximation. (b) 20 x 20 lattice approximation.

(c) 30 x 30 lattice approximation. (d) Continuous velocity field.

Figure 1: Approximations of a continuous velocity field. Adapted from Bennett and Zirbel [1].

2 Discrete Velocity Fields

Let n be an integer and let $\mathbb{D}_n = \{1, \ldots, n\}$ be an ordered set, and define a sum on $\mathbb{D}_n$ as a componentwise sum modulo n. In particular, note that $\mathbb{D}_n$ has n points, and we can identify the set $\mathbb{D}_n$ with the set of ordered counterclockwise vertices of a regular n-gon inscribed in a circle.

Definition 2.1. *A velocity field u on $\mathbb{D}_n$ is a function $u : \mathbb{D}_n \to \mathbb{Z}$ taking values on $\mathbb{Z}$. Let $\mathcal{U}$ be the set of all velocity fields u defined on $\mathbb{D}_n$. A random velocity field $\mathbf{U}$ is a stochastic process $\mathbf{U} = \{\mathbf{U}_t; t = 0, 1, \ldots\}$ taking values on $\mathcal{U}$.*

Remark 2.1. *Notice that we may represent each velocity field u as an n-dimensional vector, allowing us to write $u = [u(1)\ u(2)\ \ldots\ u(n)]$. In Figure 2, we represent two velocity fields: $u_1 = [2\ 0\ 1\ 0\ 0\ 3\ 0\ 0]$ and $u_2 = [1\ 2\ 0\ -3\ 0\ 0\ 0\ 0]$ on $\mathbb{D}_8$ using arrows. In particular, the absence of an arrow indicates that the velocity field is zero at that point.*

Remark 2.2. *Notice that each velocity field $u \in \mathcal{U}$ may act on $\mathbb{D}_n$ additively. This action can be viewed as a function $\alpha : \mathbb{D}_n \to \mathbb{D}_n$ defined by $\alpha(x) = x + u(x)$, $x \in \mathbb{D}_n$. If α corresponds to a permutation on $\mathbb{D}_n$, we say u is called incompressible; otherwise, it is called compressible.*

Two important definitions regarding the statistical properties of velocity fields are as follows:

Definition 2.2. *Let $\mathbf{U}$ be a random velocity field. For each $x_0 \in \mathbb{D}_n$, define the random velocity field $\tilde{\mathbf{U}}$ by $\tilde{\mathbf{U}}_t(x) = \mathbf{U}_t(x_0 + x)$, for $x \in \mathbb{D}_n$ and for $t = 0, 1, 2, \ldots$. We say that $\mathbf{U}$ is homogeneous if $\tilde{\mathbf{U}}$ has the same probability law as $\mathbf{U}$ for all $x_0 \in \mathbb{D}_n$.*

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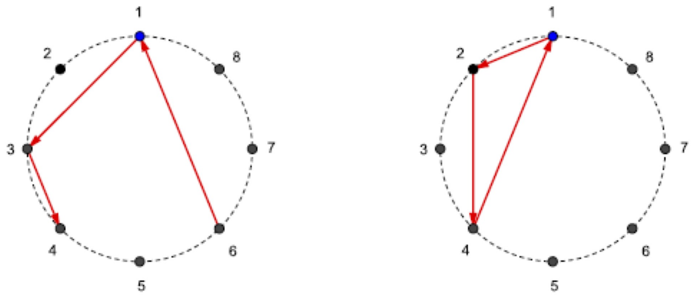

(a) Velocity field u_1. (b) Velocity field u_2.

Figure 2: Examples of two velocity fields on $\mathbb{D}_8$. Source: author.

Definition 2.3. *Let* $\mathbf{U}$ *be a random velocity field. For each* $t_0 \in \{0, 1, 2, \ldots\}$*, define the random velocity field* $\tilde{\mathbf{U}}$ *defined by* $\tilde{\mathbf{U}}_t(x) = \mathbf{U}_{t_0+t}(x)$*, for* $x \in \mathbb{D}_n$ *and for* $t = 0, 1, 2, \ldots$*. We say that* $\mathbf{U}$ *is stationary if* $\tilde{\mathbf{U}}$ *has the same probability law as* $\mathbf{U}$ *for all* $t_0 = 0, 1, 2, \ldots$*.*

The particle's trajectory in discrete space and time satisfies the following equation

$$\mathbf{X}_{t+1} = \mathbf{X}_t + \mathbf{U}_t(\mathbf{X}_t), \quad t = 0, 1, 2, \ldots; \quad \mathbf{X}_0 = \mathbf{0}, \tag{3}$$

where $\mathbf{U}_t$, for $t = 0, 1, 2, \ldots$, takes values in the set of all velocity fields on $\mathbb{D}_n$. Analogous to the continuous case, the Lagrangian velocity field $\mathbf{V}$ satisfies the equation below

$$\mathbf{V}_t(x) = \mathbf{U}_t(x + \mathbf{X}_t), \quad x \in \mathbb{D}_n, \quad t = 0, 1, 2, \ldots. \tag{4}$$

Therefore, we can write the stochastic process $\mathbf{X}_t$ as an additive functional of $\mathbf{V}$ as

$$\mathbf{X}_t = \mathbf{X}_0 + \sum_{s=0}^{t-1} \mathbf{V}_s(\mathbf{0}), \quad t = 0, 1, 2, \ldots. \tag{5}$$

Remark 2.3. *As far as we know, there are few theoretical results concerning these processes in discrete space and time. However, similar to the continuous space-time case, if* $\mathbf{U}$ *is homogeneous, stationary, and incompressible, then* $\mathbf{V}$ *is strictly stationary [1, 8].*

3 Eulerian and Lagrangian Location Processes

In this work, we consider a particular case of a random velocity field in discrete space and time, where it is described by a single incompressible vortex that moves according to a random walk. Formally, let $u \in \mathcal{U}$ be an incompressible velocity field. Let $c : \mathbb{D}_n \to [0, 1]$ be a non-negative function defined on $\mathbb{D}_n$ such that $\sum_{x \in \mathbb{D}} c(x) = 1$. Let $A_0, A_1, \ldots$ be independent and identically distributed random variables in $\mathbb{D}_n$, with $\mathbb{P}(A_t = x) = c(x)$. Let L_0 be uniformly distributed on $\mathbb{D}_n$ and independent of A_i, for $i = 0, 1, \ldots$. The Eulerian location process $\mathbf{L}$ is given by

$$L_{t+1} = L_t + A_t, \quad t = 0, 1, 2, \ldots. \tag{6}$$

Moreover, the transition matrix P of the Eulerian location process $\mathbf{L}$ is given by

$$P(y, z) = \mathbb{P}(L_{t+1} - L_t = z - y) = c(z - y), \tag{7}$$

for $y, z \in \mathbb{D}_n$. In particular, P is a doubly stochastic circulant matrix.

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Remark 3.1. *Notice that we can define a velocity field* $\mathbf{U}$ *by*

$$\mathbf{U}_t(x) = u(x - L_t), \quad x \in \mathbb{D}_n, \quad t = 0, 1, 2, \ldots. \tag{8}$$

So, $\mathbf{U}$ *is incompressible, homogeneous, and stationary [1]. Since* $\mathbf{U}_t$ *depends on the Markov process* L_t*, we can conclude that* $\mathbf{U}$ *has a hidden Markov structure.*

The generalized Lagrangian velocity $\mathbf{V}$ can be written in terms of the vortex location L_t and the particle position $\mathbf{X}_t$ as $\mathbf{V}_t(x) = u(x-(L_t-\mathbf{X}_t))$, for $x \in \mathbb{D}_n$. So, we can define a new stochastic process $\mathbf{M}$ by $M_t = L_t - \mathbf{X}_t$, $t = 0, 1, 2, \ldots$, called the Lagrangian location.

Remark 3.2. *Notice that* $\mathbf{M}$ *evolves over the time according to*

$$M_{t+1} = \sigma(M_t) + A_t, \quad t = 0, 1, 2, \ldots, \tag{9}$$

where $\sigma : \mathbb{D}_n \to \mathbb{D}_n$ *is defined by* $\sigma(x) = x - u(-x)$*, for all* $x \in \mathbb{D}_n$*. Therefore,* $\mathbf{M}$ *is Markov.*

Remark 3.3. *Comparing Eulerian and Lagrangian location processes,* $\mathbf{L}$ *and* $\mathbf{M}$*, given by Eq. (6) and Eq. (9), respectively, we observe that these two processes are not significantly different. The Eulerian process involves successive shifts due to the action of* A_t*, while the Lagrangian process develops through a shuffle* $\sigma(M_t)$ *followed by a shift* A_t*. We know the transition matrix* P *for the Eulerian location process* $\mathbf{L}$*, according to Eq. (7). Let* Q *be the transition matrix for the Lagrangian location process* $\mathbf{M}$*. Thus, we can write the transition matrix* Q *for the Lagrangian location process* $\mathbf{M}$ *as* $Q = \Sigma P$*, since* $\mathbf{M}$ *makes a deterministic transition due to the action of* σ*, which can be represented by a permutation matrix* Σ*, followed by the addition of* A_t*, or simply the action of* P*.*

3.1 Convergence to the Stationary Distribution

We can ensure that both Eulerian and Lagrangian location processes are irreducible and aperiodic by construction [1]. Since an irreducible and aperiodic Markov chain converges to its unique stationary distribution [5], and the spatial domain $\mathbb{D}_n$ is a finite lattice with u as an incompressible velocity field, both processes converge to the uniform distribution on $\mathbb{D}_n$. This leads to an interesting question: assuming the initial position of the Markov chain is known, how long does it take for the probability of finding the particle at any given position to become uniform across $\mathbb{D}_n$?

Remark 3.4. *Notice that, as a general result, if* T *is the transition matrix of a Markov chain, then the second-largest eigenvalue modulus of* T*, denoted by* $Eig_2(T)$*, when well-defined and strictly less than* 1 *in modulus, determines the rate of convergence to equilibrium [4]. Equivalently,* $Eig_2(T)$ *also controls how quickly* T^n *converges to its limit as* $n \to \infty$*. Moreover, every doubly stochastic matrix* T *has at least one eigenvalue equal to* 1*, and for an irreducible and aperiodic doubly stochastic matrix, all other eigenvalues have a modulus strictly less than* 1*.*

In fact, we are interested in reformulating the question above in terms of $Eig_2(P)$ and $Eig_2(Q)$, which define the transition matrices for the Eulerian and Lagrangian location processes, respectively, and understanding, based on the magnitude of $Eig_2(P)$ and $Eig_2(Q)$, how quickly each process converges to its invariant distribution.

3.2 Eulerian Location Process as a Random Walk

Let P be the transition matrix of the Eulerian location process $\mathbf{L}$. By defining the location process $\mathbf{L}$ as a simple random walk, we can interpret the entries of P as the probabilities of the vortex of the velocity field moving one unit to the right, to the left, or remaining in the same position; that is, $P_{ij} = \mathbb{P}(L_1 = j \mid L_0 = i)$, for $i, j = 1, 2, \ldots, n$. More specifically, the matrix P is circulant and takes the following simple form:

$$P = \begin{bmatrix} \pi_1 & \pi_2 & 0 & \cdots & 0 & \pi_n \\ \pi_n & \pi_1 & \pi_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \pi_2 & 0 & 0 & \cdots & \pi_n & \pi_1 \end{bmatrix}, \tag{10}$$

where $\pi_1, \pi_2, \pi_n \geq 0$ and $\pi_1 + \pi_2 + \pi_n = 1$. For simplicity, we use $P = [\pi_1 \;\; \pi_2 \;\; 0 \;\; \ldots \;\; \pi_n]$.

3.3 Circular and Quasi-Circular Velocity Fields

In this subsection, we introduce and define a set of circular and quasi-circular velocity fields u with a simple structure. To illustrate these velocity fields, for example, consider the spatial domain $\mathbb{D}_6$ and velocity fields u_i, for $i = 1, 2, \ldots, 6$, as shown in Figure 3.

Remark 3.5. *Note that each velocity field u_i represented in Figure 3 is incompressible, allowing us to associate a permutation matrix Σ_i with it. From now on, we will identify the velocity field u_i and the permutation matrix Σ_i as representing the same vortex type. For each velocity field u_i, we can assign its size as the number of consecutive points moved one unit counterclockwise by u_i. For example, velocity field identified by Σ_4 has size* 3.

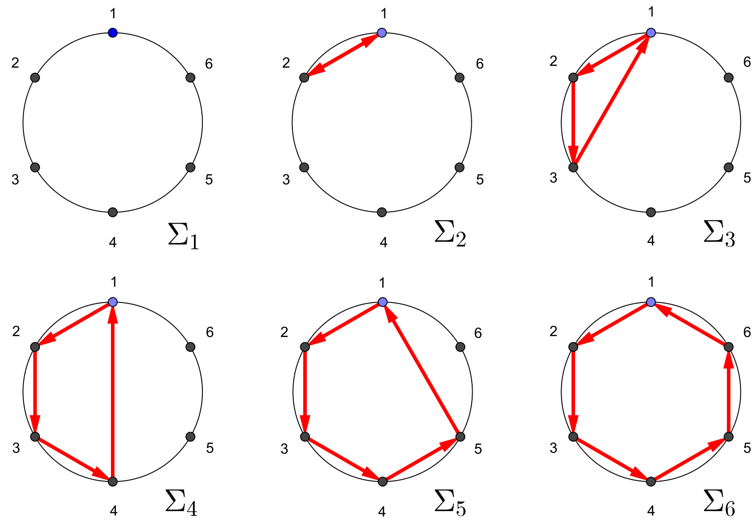

Figure 3: Circular and quasi-circular velocity fields in $\mathbb{D}_6$. Source: Author.

We aim to analyze the behavior of $Eig_2(\Sigma_i P)$, which provides insight into the rate of convergence to the invariant distribution of the Lagrangian location process $\mathbf{M}$.

Remark 3.6. *Let $P = [0.2 \;\; 0.3 \;\; 0 \;\; \ldots \;\; 0 \;\; 0.5]$ be a circulant matrix, associated with the Eulerian location process $\mathbf{L}$ described by Eq.* (6)*. Thus, the transition matrix P assigns probabilities of* 0.5 *to move left,* 0.3 *to move right, and* 0.2 *to stay in place. Let $Q_i = \Sigma_i P$, for $i = 1, 2, \ldots, 6$, be defined according to the corresponding velocity fields shown in Figure 3, which are associated with the Lagrangian location process $\mathbf{M}$ as in Eq.* (9)*. Notice that Q_i does not have the same circulant* *structure as P, except for the matrix Σ_1. In fact, Σ_i permutes some rows of P to obtain the matrix Q_i. We calculated the second-largest eigenvalue modulus $Eig_2(\Sigma_i P)$ for each matrix Σ_i, for $i = 1, 2, \ldots, 6$, and plotted a graph as shown in Figure 4. This allows us to explore graphically the behavior of $Eig_2(Q_i)$ as we increase the size of the velocity field u_i. We highlight the structure of $Q_4 = \Sigma_4 P$, where we found the minimum second-largest eigenvalue modulus. This indicates that the random velocity field associated with u_4 converges to the invariant distribution more quickly than the other circular or quasi-circular velocity fields on $\mathbb{D}_6$.*

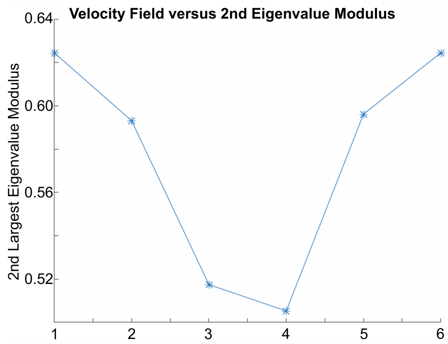

Figure 4: 2^{nd} largest eigenvalue modulus versus size of Σ_i for $n = 6$. Source: Author.

That analysis also raises an important question about whether the spatial domain size is a determining factor in the location of the second-largest eigenvalue modulus and how the size of the spatial domain affects the rate of convergence in such settings. In Figure 5, we consider two distinct domains for $n = 40$ and $n = 41$ while keeping the same transition matrix P. These graphs suggest that for larger spatial domains, we observe slower convergence to the invariant distribution. Moreover, we observe a slight variation in the behavior of the graphs when considering the parity of the spatial domains: for $n = 40$, the velocity field u_{21} provides the highest convergence rate; for $n = 41$, velocity fields u_{21} and u_{22} have similarly fast convergence rates.

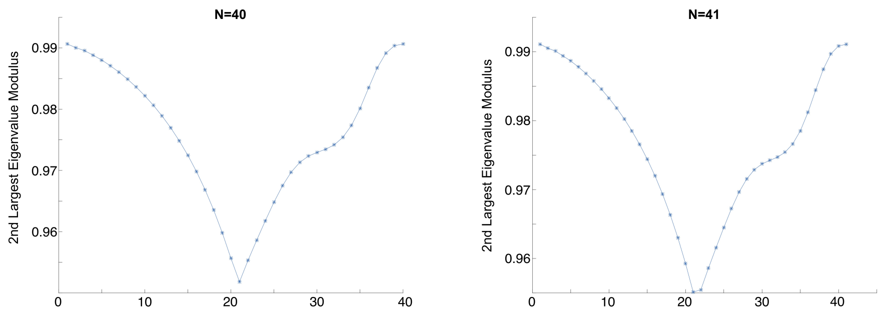

Figure 5: 2^{nd} largest eigenvalue modulus versus size of Σ_i for P constant. Source: Author.

Remark 3.7. *By considering distinct transition matrices P, associated with different random walks on $\mathbb{D}n$, we can also explore the effect of P on convergence to equilibrium. In Figure 6, we consider two different matrices P and observe distinct patterns in the resulting graphs, even for the same spatial domain $\mathbb{D}_{40}$ and velocity fields, across all possible circular or quasi-circular velocity fields. This observation raises theoretical questions on how random walks affect convergence rates.*

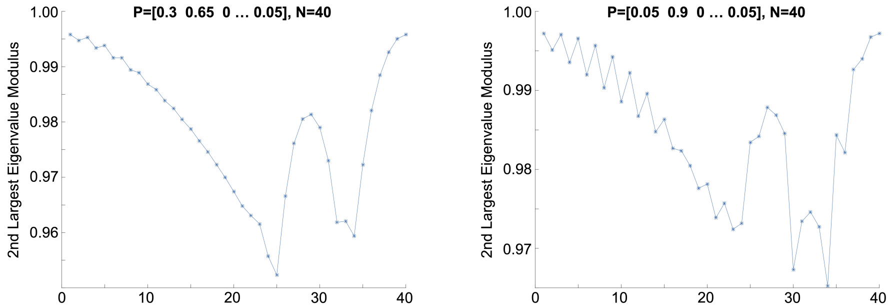

Figure 6: 2^{nd} largest eigenvalue modulus versus size of Σ_i for distinct matrices P. Source: Author.

4 Conclusion

In this work, we propose a random velocity field on one-dimensional finite lattices with periodic boundary conditions and discrete-time dynamics. We define the Eulerian and Lagrangian location processes, examining their interdependence. By modeling the Eulerian process as a random walk governed by a circulant matrix P and circular or quasi-circular velocity fields represented by permutation matrices Σ_i, we analyze the convergence rate of the Lagrangian process to its invariant distribution, tied to the second-largest eigenvalue modulus of matrices $Q_i = \Sigma_i P$.

Beyond our analysis, we explore how spatial domain size, parity, and different choices of P influence convergence to equilibrium. These findings suggest further research into how structural variations in discrete velocity fields affect spectral properties of Eulerian and Lagrangian processes. Future work may extend this approach to higher-dimensional lattices, non-circulant velocity fields, or alternative transition structures, broadening our understanding of convergence behaviors in discrete stochastic systems.

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