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Probability Punishment in Spatial Optional Public Goods Games

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In Optional Public Goods Games, at each round time, players can adopt three possible strategies: cooperate, defect, or abstain (loners). Free-riders pose a threat to cooperation even in the absence of loners. To counter this, incentive mechanisms like decentralized (*peer punishment*) and centralized (*pool punishment*) sanctions are normally implemented [1, 4]. Following the methodology introduced in [2], where a peer punishment mechanism modifies the strategy payoffs by incorporating a cost of punishment to cooperators and reducing the benefits of defectors, we introduce a probability P_0 determining whether each cooperator becomes a punisher, imposing sanctions on defectors. Specifically, the decision to punish is determined by a Bernoulli random variable $X \sim \text{Bernoulli}(P_0)$. More precisely, on one hand, being a punisher cooperator has a cost α for each defector sanctioned, this situation reduces its payoff, on the other hand, if a cooperator is not selected to be a punisher they receive their normal payoff. Punished defectors have their payoff reduced by a fine β imposed by each punishing cooperator, where $\alpha < \beta$. Therefore, the payoffs are defined as follows:

$$P_c = \frac{rcn_c}{n} - c; \quad P_{cp} = \frac{rcn_c}{n} - c - \alpha\, n_d; \quad P_{dp} = \frac{rcn_c}{n} - \beta\, n_{cp}; \quad P_l = \sigma, \tag{1}$$

where n_i with $i \in \{c, d, l\}$ corresponds to the number of cooperators, defectors and loners; $n = n_c + n_d$, r is the rate of interest and c is the contribution to the game. P_c is the payoff of the cooperator, when a cooperator is selected to punish, with P_{cp} being the payoff of the punishing cooperator, P_{dp} the payoff of the punished defector, n_d is the number of defectors sanctioned, and n_{cp} is the number of punishing cooperators imposing the fine. $P_l = \sigma$ is the constant payoff received by loners.

The simulation was implemented on a 50×50 grid, where each cell can adopt one of three strategies: Loners, Cooperators, or Defectors. At each step, the fitness of each cell is calculated based on its strategy and that of its neighbors, and the strategies are updated with a given probability of 0.8. In the experiments, the key parameters were fixed: multiplication factor $r = 2.2$, strategy adoption probability 0.8 based on [3], and punishment values $\alpha = 1.0$ and $\beta = 1.5$ following [2], where $\beta > \alpha$ ensures dynamic equilibrium. The experiments were run for 5000 iterations, sufficient to analyze trajectories and compare P_0 values; all snapshots correspond to the system state at iteration 1000 for direct comparison across cases.

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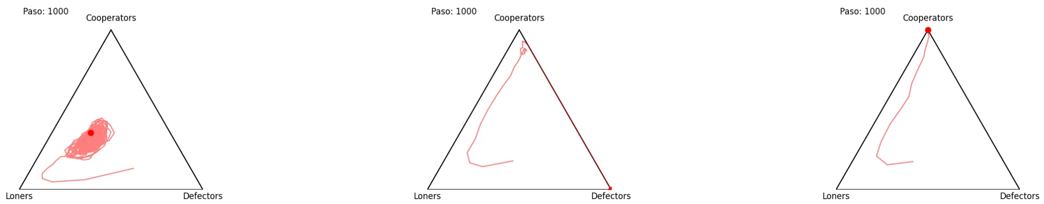
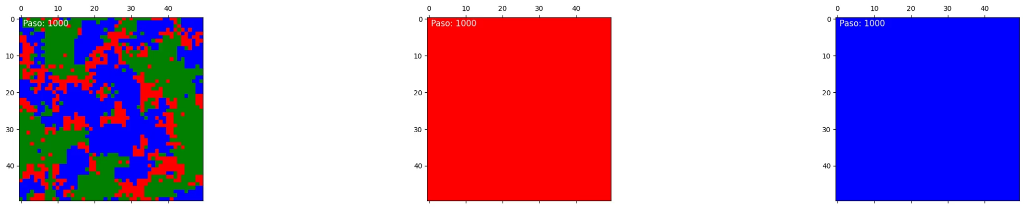


(a) $P_0 = 0.0$. (b) $P_0 = 0.17$. (c) $P_0 = 0.25$.



(d) $P_0 = 0.0$. (e) $P_0 = 0.17$. (f) $P_0 = 0.25$.

Figure 1: Experimental results of the effect of different punishment values. (a-c) Simplex Diagrams and (d-f) Lattice Diagrams showing system dynamics for $P_0 = 0.0$, 0.17 and 0.25 respectively. Source: Authors own work.

The results are presented in Figure 1. In (a) $P_0 = 0.0$, the behavior exhibits a "Red Queen Mechanism" [3]. The strategies co-exist cyclically. (b) $P_0 = 0.17$: At this critical threshold, loners are extinct, breaking the stabilizing cycle and enabling defector dominance due to its higher fitness. Stochastic updates reinforce this dominance, showing that $P_0 = 0.17$ fails to effectively promote cooperation. (c) $P_0 = 0.25$: The system fully converges to cooperation, eliminating cycles. Punishment suppresses defectors effectively, showing that high rates are unnecessary, as punishers and loners extinction enforce cooperation naturally.Numerical experiments show the role of punishment probability in the global behavior, enhancing cooperation. Ongoing research explores the effects of population size and payoff parameters on convergence and stability.

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