Proceeding Series of the Brazilian Society of Computational and Applied Mathematics, v. 12, n. 1, 2026.

Trabalho apresentado no XLIV CNMAC, Fundação Getulio Vargas - Rio de Janeiro - RJ, 2025.

Proceeding Series of the Brazilian Society of Computational and Applied Mathematics

Some Additional Properties on Gaussian Quadrature Rules Obtained from Quasi-Symmetric Orthogonal Polynomials

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Recently, quasi-symmetric orthogonal polynomials and associated Gaussian quadrature rules were studied in [3]. It is well known that orthogonal polynomials on the real line satisfy a three-term recurrence relation (see [1]). In the quasi-symmetric case, the orthogonal polynomials satisfy the following three-term recurrence relation

$$P_{n+1}^{\phi}(x) = (x - \beta_{n+1}^{\phi})P_n^{\phi}(x) - \gamma_{n+1}^{\phi}P_{n-1}^{\phi}(x), \quad n \geq 1, \tag{1}$$

with $P_0^{\phi}(x) = 1$, $P_1^{\phi}(x) = x - \beta_1^{\phi}$, $\beta_{2n-1}^{\phi} = a$, $\beta_{2n}^{\phi} = b$, $a, b \in \mathbb{R}$, and $\max\{|a|, |b|\} > 0$. Here, γ_{n+1}^{ϕ} is a positive real number given in terms of the associated orthogonality measure ϕ (see [3, Eq. (2)]).

Now, if η is the symmetric orthogonality measure for orthogonal polynomials obtained when $a = b = 0$ in (1) and ϕ is the orthogonality measure corresponding to the case where $a, b \in \mathbb{R}$, with at least a (or b) different from zero, then the measures ϕ and η are related by (see [3, Theorem 4])

$$d\phi(x) = \frac{2(x-b)}{|p'_{a,b}(x)|} d\eta(\sqrt{p_{a,b}(x)}\,),$$

with $supp(\phi) \subset I_a^b = (-\infty, m] \cup [M, \infty)$, $m = \min\{a, b\}$, $M = \max\{a, b\}$, and $p_{a,b}(x) = (x - a)(x - b)$.

In [3], the complete characterization of quasi-symmetric orthogonal polynomials was given in terms of the associated symmetric ones. In the same paper, connection formulas for the weights and nodes in the corresponding Gaussian quadrature rules were also established. In fact, for $n \geq 1$, it was shown that the nodes in the associated Gaussian quadrature rule are connected by

$$x_{2n,k}^{\phi} = c_k \sqrt{(x_{2n,k}^{\eta})^2 + \left(\frac{a-b}{2}\right)^2} + \left(\frac{a+b}{2}\right), \qquad 1 \leq k \leq 2n,$$

and

$$x_{2n+1,k}^{\phi} = \begin{cases} c_k \sqrt{(x_{2n+1,k}^{\eta})^2 + \left(\frac{a-b}{2}\right)^2} + \left(\frac{a+b}{2}\right), & 1 \leq k \leq 2n+1, \quad k \neq n+1, \\ a\,, & k = n+1. \end{cases}$$

Moreover, the weights are related by

$$w_{2n,k}^{\phi} = \left(\frac{c_k \sqrt{(x_{2n,k}^{\eta})^2 + \left(\frac{a-b}{2}\right)^2} + \left(\frac{a-b}{2}\right)}{c_k \sqrt{(x_{2n,k}^{\eta})^2 + \left(\frac{a-b}{2}\right)^2}} \right) w_{2n,k}^{\eta}\,, \quad 1 \leq k \leq 2n,$$

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and

$$w^{\phi}_{2n+1,k} = \begin{cases} \left(\frac{c_k \sqrt{(x^{\eta}_{2n+1,k})^2 + \left(\frac{a-b}{2}\right)^2} + \left(\frac{a-b}{2}\right)}{c_k \sqrt{(x^{\eta}_{2n+1,k})^2 + \left(\frac{a-b}{2}\right)^2}} \right) w^{\eta}_{2n+1,k}, & 1 \leq k \leq 2n+1,\ k \neq n+1, \\ w^{\eta}_{2n+1,k}, & k = n+1. \end{cases}$$

Here, $c_k = -1$ if $k \in \{1, 2, \ldots, n\}$ and $c_k = 1$, otherwise.

Thus,

$$\int_{I_a^b} f(x) d\phi(x) = \sum_{k=1}^{n} w^{\phi}_{n,k} f(x^{\phi}_{n,k}) + E_n(f), \tag{2}$$

where $E_n(f)$ is the error in the approximation (see [2]).

In this work, for $b > 0$, and considering $a = -b$, we show that when f is an even function defined in I_a^b the weights in (2) can be simply replaced by $w^{\eta}_{n,k}$. Moreover, we also show that the error in the approximation is preserved. Thus,

$$\int_{I_a^b} f(x) d\phi(x) = \sum_{k=1}^{n} w^{\eta}_{n,k} f(x^{\phi}_{n,k}) + \hat{E}_n(f), \tag{3}$$

with $\hat{E}_n(f) = E_n(f)$, which means that we can use the weights associated with the symmetric case in order to approximate the quasi-symmetric one. Notice that the nodes remain the same in (2) and (3).

We believe that this result can be extended for any general function f defined in I_a^b through the well-known decomposition

$$f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2} = \left(\frac{f(x) + f(-x)}{2} \right) + x \left(\frac{f(x) - f(-x)}{2x} \right).$$

Notice that $f(x) = g(x) + xh(x)$, where g and h are even functions. Then, from the results obtained in [3, Lemma 1], the odd part of f can be calculated in terms of the parameter b and the even function h. Another subject of future interest is the analysis of the error when f is a general function. Taking into account our initial results, it seems that the error will necessarily depend on the parameter b as well as the behavior of the error in the purely symmetric case. Finally, we also intend to obtain similar results for the case $a \neq -b$.

Acknowledgment

The authors thank the Federal University of Triângulo Mineiro for supporting the presentation of this work at CNMAC 2025.

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