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Random Walks Determined by a Class of Generalized Fibonacci polynomials

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In this work, we briefly review the definitions of random walks and birth-and-death processes, as these topics are well established. For further details, the interested reader is referred to Dominguez [3] and references therein. These stochastic processes constitute a special class of Markov processes with a discrete state space.

We derive the one-step transition matrices for these two types of Markov processes in discrete time. With these matrices at hand, we aim to compute their corresponding n-step transition probabilities, which are given by the entries of the n–th power of the one–step transition matrix. A successful method for this computation is the Karlin-McGregor representation; see Karlin and McGregor [4] for a discussion on the relationship between orthogonal polynomials and random walks. Our interest in studying orthogonal polynomials lies in understanding which random walks are induced by such polynomials (see, for instance, [1]). In this work, we are particularly interested in determining which generalized Fibonacci polynomials (see, for instance, [2]) induce or characterize a random walk. We begin by summarizing key concepts introduced by Flórez et al. [2] for Generalized Fibonacci Polynomials (GFP). Specifically, we consider two fixed nonzero polynomials, $d(x)$ and $g(x)$ in $\mathbb{Q}[x]$ with $\deg(d(x)) > \deg(g(x))$. For $n \geq 2$, a second-order polynomial recurrence relation of **Fibonacci-type** is defined as follows:

$$\mathcal{F}_0(x) = 0, \quad \mathcal{F}_1(x) = 1, \quad \text{and} \quad \mathcal{F}_n(x) = d(x)\mathcal{F}_{n-1}(x) + g(x)\mathcal{F}_{n-2}(x). \tag{1}$$

Similarly, a second-order polynomial recurrence relation of **Lucas-type** satisfies the relation:

$$\mathcal{L}_0(x) = p_0, \quad \mathcal{L}_1(x) = p_1(x), \quad \text{and} \quad \mathcal{L}_n(x) = d(x)\mathcal{L}_{n-1}(x) + g(x)\mathcal{L}_{n-2}(x), \tag{2}$$

where $p_0 \in \{\pm 1, \pm 2\}$ and $p_1(x)$, $d(x) = \alpha p_1(x)$, and $g(x)$ are fixed non-zero polynomials in $\mathbb{Q}[x]$ with α an integer of the form $2/p_0$. Using the special case of Favard's Theorem, we examine the orthogonality of GFPs. We investigate sufficient conditions on the set of parameters under which the induced family of polynomials is not only of Lucas type but also belongs to well-known families of orthogonal polynomials, such as the Chebyshev polynomials of the first kind and the Morgan-Voyce polynomials.

Since the generalized Fibonacci polynomials are orthogonal, we will determine the conditions under which they qualify as random walk polynomials. To this end, we use the definition random walk polynomial sequence and the following theorem:

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Definition 1. *A polynomial sequence* $(P_n(x))$ *that is orthogonal with respect to a measure on* $[-1,1]$ *and for which the parameters* α_n *in the recurrence relation*

$$\begin{aligned} P_{n+1}(x) &= (x-\alpha_n)P_n(x) - \beta_n P_{n-1}(x),\ n \geq 1, \\ P_0(x) &= 1,\ P_1(x) = x - \alpha_0, \end{aligned} \tag{3}$$

are nonnegative is called a ***random walk polynomial sequence****.*

Any measure with respect to which a random walk polynomial sequence is orthogonal is called a ***random walk measure****.*

Theorem 1 ([1])**.** *The following statements are equivalent:*

1. *The sequence* $(P_n(x))$ *is a random-walk polynomial sequence.*
2. *There are numbers* $p_n > 0$*,* $q_{n+1} > 0$*, and* $r_n \geq 0$ *for* $n \geq 0$ *satisfying* $p_0 + r_0 \leq 1$ *and* $p_n + q_n + r_n = 1$ *for* $n \geq 1$*, such that* $\alpha_n = r_n$ *and* $\beta_{n+1} = p_n q_{n+1}$ *for* $n \geq 0$*.*
3. *The sequence* $(P_n(x))$ *is orthogonal with respect to a measure with support in* $[-1,1]$ *and satisfies* $\alpha_n \geq 0$ *for* $n \geq 0$*.*

Discrete–time random Walks determined by a class of generalized Fibonacci polynomials

The following proposition requires that:

- The GFP of Lucas type, $\mathcal{L}_n(x)$, is defined with initial conditions $\mathcal{L}_0(x) = p_0$, $\mathcal{L}_1(x) = p_1(x)$ as given in (2).
- $d(x) = cx + h$ and $g(x) = -(c-1+h)$ where $c, h \in \mathbb{Z}$, $h \leq 0$, and $c > 1-h > 0$.

Proposition 1. *Let* $\omega(x) := \sqrt{4(c-1+h)-(cx+h)^2}$*. If*

$$\frac{-\sqrt{4(c-1+h)-1}-h}{c} \leq x \leq \frac{\sqrt{4(c-1+h)}-h}{c}, \tag{4}$$

then $1/\omega(x) \subset [-1,1]$*. Moreover,* $\mathcal{L}_n(x)$ *determines a random walk.*

In this work we address both, discrete and continuous–time, nearest neighbor random walks.

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