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New Properties for Markov Evolution Algebras

José M. Toro[1], Pablo M. Rodriguez[2]
UFPE, Recife, PE
Irene Paniello[3]
UNIZAR, Saragoça, ESPANHA

Evolution algebras are a class of non-associative algebras that arise as a mathematical model to represent non-Mendelian genetics. They have sparked great interest in various fields of knowledge such as graph theory, dynamical systems, and Markov chains. The concept was first introduced by [4], in a finite-dimensional approach, and later extended by [3].

Definition 1. *[3, Definition 3] Let Λ be a countable set. An evolution algebra $\mathcal{A}$ is an $\mathbb{R}$-algebra that admits a natural basis $\mathcal{B} = \{\boldsymbol{e}_i \mid i \in \Lambda\}$, such that $\boldsymbol{e}_i \cdot \boldsymbol{e}_i = \sum_{k\in\Lambda} c_{ik}\boldsymbol{e}_k$, for all $i \in \Lambda$, and $\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \mathbf{0}$, for all $i, j \in \Lambda$ such that $i \neq j$.*

Whether $c_{ik} \in [0, 1]$, for any $i, k \in \Lambda$, and $\sum_{k\in\Lambda} c_{ik} = 1$, for any $i \in \Lambda$, then $\mathcal{A}$ is called a Markov evolution algebra. Thus defined, $\{c_{ik}\}$ are transition probabilities of a discrete-time Markov chain, so a correspondence can be established between $\mathcal{A}$ and a discrete-time Markov chain $(X_n)_{n\geq 0}$ with state space $S = \{x_i \mid i \in \Lambda\}$ and transition probabilities given by $c_{ik} := \mathbb{P}(X_{n+1} = x_k|X_n = x_i)$, for $i, k \in \Lambda$ and $n \in \mathbb{N}$. In other words, a Markov evolution algebra $\mathcal{A}$ with a natural basis $\mathcal{B}$ "generates" a discrete-time homogeneous Markov chain with time space S. The first reference discussing the interplay between evolution algebras and Markov chains is [3, Chapter 4], where many well-known results coming from Markov chains are stated in the language of Markov evolution algebras. In this work, we review the properties established by [3] and, by exploring their connection with Probability Theory, simplify some of the proofs. In addition, we extend some results and establish new properties for the case where Λ is not finite. The following results are part of [5].

Theorem 1. *Let $\{X_n\}_{n\geq 0}$ be a Markov chain with countable state space S, such that:*

$$| \{k \in S : c_{ik} > 0\} | < \infty, \text{ for all } i \in S. \tag{1}$$

Then, $\{X_n\}_{n\geq 0}$ generates a Markov evolution algebra $\mathcal{A}(X_n)$ with natural basis $\mathcal{B} = \{\boldsymbol{e}_i \mid i \in S\}$ and structure constants given by $c_{ik} := \mathbb{P}(X_{n+1} = k|X_n = i)$, for $i, k \in S$ and for any $n \in \mathbb{N}$.

Theorem 1 formalizes how a Markov evolution algebra is associated with a given discrete-time Markov chain, provided that (1) holds for any $i \in S$. This result extends and improves [3, Theorem 16], because although that theorem claims that any discrete-time Markov chain can be associated with a Markov evolution algebra, this is not entirely true when S is a countably infinite state space and (1) is not taken into account. For counterexamples, see [5, Section 2.1] and [1, Example 1]. This is because, in [3, Definition 3], the basis is implicitly assumed to be a Hamel basis. Once this connection is well-established, certain properties can be derived by exploring the relationship between Markov chains and evolution algebras. In what follows, we state one of these properties and refer the reader to [5] for further details, and additional results.

[1]jose.jaramillo@ufpe.br
[2]pablo@de.ufpe.br
[3]ipaniello@posta.unizar.es

Theorem 2. *Let $\mathcal{A}(X_n)$ be the Markovian evolution algebra generated by the Markov chain $\{X_n\}_{n\geq 0}$, and let $S_0 \subsetneq S$. S_0 is a closed class in the Markov chain if, and only if, span$\{\mathbf{e}_i \mid i \in S_0\}$ is a simple evolution subalgebra.*

Theorem 2 is related to [3, Theorem 17] which states that S_0 is a closed subset in S if and only if *span*$\{\mathbf{e}_i \mid i \in S_0\}$ is an evolution subalgebra. Our result gains in interest if we recognize that understanding the closed classes of a Markov chain is essential for analyzing its long-term behavior.

Exemple 1. *Let $\{X_n\}_{n\geq 0}$ with digraph of transitions given by Figure 1 and consider $\mathcal{A}(X_n)$. The classes are $C(1)$, $C(3)$, $C(4)$. In the evolution algebra, $\langle \mathbf{e}_1 \rangle = span\{\mathbf{e}_1, \mathbf{e}_2\}$, $\langle \mathbf{e}_3 \rangle = span\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. If $S_0 = \{1, 2, 3\}$, a closed subset that is not a class, then span$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is an evolution subalgebra that is not simple. If $S_0 = \{1, 2\}$, a closed class, then span$\{\mathbf{e}_1, \mathbf{e}_2\}$ is a simple evolution subalgebra.*

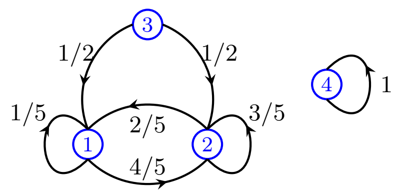

Figure 1: Digraph of transitions for the Markov chain of Example 1. Source: Authors.

This work explores the connection between Markov chains and evolution algebras, where Markov chain concepts are expressed in a non-associative algebraic framework. After reviewing recent results from [2, 3], we simplify some proofs and derive new properties.

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