Analysis of Composite Rules of Classes II and II from Wolfram's Elementary Cellular Automata

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Cellular automata are mathematical models used to simulate dynamic systems where time is discrete and state space is finite, evolving over time. They consist of a matrix, or grid, of cells, each of which has a state belonging to a finite set of alphabets, such as $\{0, 1\}$. Each cell evolves according to rules that depend solely on its individual state and the state of a finite number of neighboring cells. It is important to note that, in most cases, the evolution of a cellular automaton is irreversible, as multiple sets of states can lead to the same evolution. In his research, Wolfram classified *Elementary Cellular Automata* (ECA), the simplest type of cellular automaton, into four classes. In this work, the class II composite ECA are studied and analyzed in relation to the Wolfram classification and all the codes used were developed in the Python programming language [3].

A *cellular automaton* is a quintuple $C = (\mathcal{L}, \mathcal{S}, c, n, \mathcal{R})$ where $\mathcal{L}$ represents the size of the set of cells, also known as *grid*. Similarly, $\mathcal{S}$ is a finite set of states, such as $\{0, 1\}$. The parameter c defines the *initial configuration*, which is a specific mapping of states to the cells in the grid in the initial moment, and the parameter n specifies the size of the *neighborhood*, which consists of the number of adjacent cells to be considered. Finally, $\mathcal{R}$ is a *local rule*, defined as a function $\mathcal{R} : \mathcal{S}^{2n+1} \to \mathcal{S}$ [2].

The *configuration* c of a cellular automaton is a mapping $c : \mathbb{Z}^d \to \mathcal{S}$ that specifies the state of each cell in the grid and $\mathcal{C}$ is the set of all possible configurations. If c is constant, assigning the same state to all cells, it is called a *trivial configuration.* The local rules induce a *global transition function* $F : \mathcal{C} \to \mathcal{C}$ which is a dynamical system where, $F(c) = e$ represents a new configuration in $\mathcal{C}$. The group of cellular automata where $d = 1$, $n = 1$, and $|\mathcal{S}| = 2$ is what Wolfram called Elementary Cellular Automata. The focus of this work is exclusively on elementary models, where state 0 represents white cells, and 1 represents black cells. Let F_1 and F_2 be two global transition functions with respective local rules $\mathcal{R}_1$ and $\mathcal{R}_2$ of ECA given in the same configuration space. The composition $F_1 \circ F_2$ is also a global transition function. We study *compositions of ECA* of this type [1, 2].

In this work, we considered the composition[2] of two rules, with the following characteristics: $\mathcal{L} = 100$, $\mathcal{S} = \{0, 1\}$, c is random, using 20 different configurations, $n = 1$, and $\mathcal{R}$ is a composition whose local rules are an set of Wolfram's Class II rules, taken in pairs. As a main result, the following theorem is achieved.

Theorem 1. The compositions of elementary cellular automata from Wolfram's Class II with Class II exhibit dynamic behaviors that can align with any of the four Wolfram classes.

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