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On the Connection between Contextuality and Semidefinite Programming

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Quantum theory is one of the most prominent physical theories today, but it exhibits several "strange" properties, one of which is contextuality. This property states that the result of a measurement depends on the context in which it is being performed, specifically on the set of other possible results for the same measurement. Recently, A. Cabello highlighted a connection between contextuality and some convex sets from the theory of semidefinite programming, what is now known as the graph theoretic approach to contextuality [3]. Our work aims to further explore this connection and raise new questions.

Two events are said to be exclusive if they cannot occur simultaneously, meaning there exists a measurement for which the events yield different outcomes. Based on this, we can construct a graph G, called the graph of exclusivity. The vertex set of G represents the set of events, and an edge connects two vertices if and only if the corresponding events are exclusive.

What are the possible results of an experiment? Starting from an initial state, we can obtain a vector $p \in \mathbb{R}^{V(G)}$, called a behavior for G, where p_i is the probability of event $i \in V(G)$ occurring. The set of all possible behaviors depends on the physical theory being assumed. For example, one can demonstrate that contextuality is a property of reality by constructing an experiment that produces a behavior outside the set permitted by non-contextual theories [4]."

A. Cabello showed [3] that the set of behaviors allowed by classical theory, quantum theory, and the E-Principle (to be explained further) are, respectively, the following convex bodies, well known in the theory of semidefinite programming.

$$\text{STAB}(G) = \text{ConvHull}\{\mathbf{1}_S : S \text{ is an independent set in } G\} \tag{1}$$

$$\text{TH}(G) = \left\{x : \begin{pmatrix} 1 & x \\ x^T & X \end{pmatrix} \succcurlyeq 0, X_{ij} = 0 \ \forall\ ij \in E(G), X_{ii} = x_i \ \forall\ i \in V(G)\right\} \tag{2}$$

$$\text{QSTAB}(G) = \left\{y : \mathbf{1}_C^T y \leq 1 \text{ for every clique } C\right\} \tag{3}$$

One interesting consequence of this arises from V. Chvátal's characterization of perfect graphs [2]: a graph is perfect if and only if $\text{STAB}(G) = \text{TH}(G) = \text{QSTAB}(G)$. Thus, if one aims to construct an experiment to observe some contextual behavior, the corresponding graph of exclusivity must be imperfect. Otherwise the set of quantum and classical behaviors coincides.

The E-Principle (EP for short) states that if $S \subseteq V(G)$ is a set of mutually exclusive behaviors, then $\mathbf{1}_S^T p \leq 1$ for any behavior. Let $X(G)$ be the set of behaviors allowed in reality. In our work, we aim to understand $X(G)$, and search for some physical intuitive principles (just like the EP) which would imply $X(G) = \text{TH}(G)$, thereby providing a justification of quantum theory.

B. Amaral [1] showed that if we assume quantum theory as a principle, i.e., $\text{TH}(G) \subseteq X(G)$, then the EP ensures $\text{TH}(G) = X(G)$. Our research explores whether this can be done under weaker assumptions. For instance, it can be observed that it if H is a subgraph of G, then $X(H)$ is equal

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to the set of behaviors in $X(G)$ with support on $V(H)$. Can $\mathrm{TH}(G)$ be reconstructed by taking convex combinations of behaviors from its subgraphs? Every perfect graph with at least one edge satisfies this property. However, it fails for minimally imperfect graphs. So the following question is motivated: is it true that if G is imperfect but not minimally imperfect, then $\mathrm{TH}(G)$ can be reconstructed? We answer this question in the negative.

The EP, when applied to a single graph, gives $X(G) = \mathrm{QSTAB}(G)$. However, it is reasonable to extend the principle to scenarios where experiments are being performed simultaneously. Let G and H be graphs of exclusivity for two experiments conducted independently, and let $G * H$ denote the graph with vertex set $V(G) \times V(H)$, and edges between (g_1, h_1) and (g_2, h_2) exactly when $g_1 \sim_G g_2$ or $h_1 \sim_H h_2$. With this, the EP can be reinstated as:

$$\mathrm{ConvHull}\left[X(G) \times X(H)\right] \subseteq \mathrm{QSTAB}(G * H) \tag{4}$$

It can be verified that $X(G) = \mathrm{QSTAB}(G)$ does not satisfy the above, making equation 4 stronger then the original EP. However, at first hand, it is not immediately clear whether this is weaker or stronger than the quantum upper bound. With this in mind, we prove the following theorem:

Theorem 1. *For any graphs G, H:*

$$\mathrm{TH}(G) \times \mathrm{TH}(H) \subseteq \mathrm{TH}(G * H) \subseteq \mathrm{QSTAB}(G * H) \tag{5}$$

An immediate consequence of this theorem is that the requirement $X(G) \subseteq TH(G)$ for all G is stronger than the condition imposed by equation 4.

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