Proceeding Series of the Brazilian Society of Computational and Applied Mathematics, v. 12, n. 1, 2026.

Trabalho apresentado no XLIV CNMAC, Fundação Getulio Vargas - Rio de Janeiro - RJ, 2025.

Proceeding Series of the Brazilian Society of Computational and Applied Mathematics

Impact of Carrying Capacity on Species Competition in Optimization Strategies for Impulsive Releases

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Abstract. This work investigates the influence of the carrying capacity on the replacement of species S_1 by species S_2, with a focus on minimizing intervention costs. The problem is formulated as an optimal control problem, in which the control actions correspond to impulsive releases of individuals from species S_2. The objective is to determine optimal release strategies that ensure the eradication of S_1 and the successful establishment of S_2. Numerical results are provided to illustrate the control strategies and to analyze how the carrying capacity affects the optimal release policy.

Keywords. Optimal Control, Carrying Capacity, Impulsive Releases, Population Dynamics, Species Competition.

1 Introduction

The carrying capacity of an environment represents the maximum number of individuals that a population can sustain over time, determined by factors such as resource availability, space, and ecological interactions. In practice, however, each species experiences an effective carrying capacity, which accounts not only for environmental limitations but also for interspecific interactions, physiological traits, and adaptive mechanisms that influence how efficiently each population utilizes available resources [6]. This concept plays a crucial role in population dynamics and species competition, directly influencing which species can establish and thrive in a given habitat [7]. In many ecological and environmental management scenarios, it becomes necessary to control population equilibrium to promote the replacement of one species by another, whether for conservation purposes, invasive species control, or ecosystem optimization [8].

In this work, we investigate the influence of the carrying capacity on the replacement of species S_1 by species S_2, with a focus on minimizing intervention costs. The problem is formulated within an optimal control framework, in which control actions consist of impulsive releases of individuals from species S_2. The objective is to determine optimal release strategies that ensure the eradication of S_1 and the successful establishment of S_2 in an efficient manner. In this context, we consider effective carrying capacities (K_1 and K_2), which reflect the actual ability of each species to exploit available resources and maintain its population, rather than a uniform environmental limit. Thus, we aim to understand how the carrying capacity affects species competition and to optimize the resources required to achieve the desired replacement.

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The paper is structured as follows. In Section 2, we introduce the optimal control problem, present the cost functional to be minimized, and discuss the impulsive competition model proposed in [1]. Section 3 presents the numerical results, which illustrate the control strategies and highlight the influence of the carrying capacity on the optimal release policy. Finally, Section 4 provides concluding remarks, summarizing the main findings and outlining directions for future research.

2 Impulsive Optimal Control Problem

Consider the following impulsive differential equation model, which describes the competitive dynamics between two species, S_1 and S_2. This model was proposed in [1], captures the effects of ecological interactions and discontinuous perturbations in the system.

$$\begin{cases} \dfrac{dS_1(t)}{dt} = S_1(t)\left(\psi_1 - \dfrac{r_1}{K_1}(S_1(t)+S_2(t))\right)\left(\dfrac{S_1(t)}{K_0} - 1\right) - \delta_1 S_1(t), \\ \dfrac{dS_2(t)}{dt} = S_2(t)\left(\psi_2 - \dfrac{r_2}{K_2}(S_1(t)+S_2(t))\right) - \delta_2 S_2(t), \end{cases} \quad \text{if } t \neq n\tau,\ n = 1,2,...,N. \tag{1}$$

$$\begin{cases} S_1(t^+) = S_1(t), \\ S_2(t^+) = S_2(t) + u_n, \end{cases} \quad \text{if } t = n\tau,\ n = 1,2,...,N. \tag{2}$$

The model is considered with nonnegative initial conditions and positive parameters. For $i = 1, 2$, ψ_i and δ_i represent the birth and mortality rates of species S_i, respectively, while $r_i = \psi_i - \delta_i$ denotes the intrinsic growth rate. In addition, K_i is a parameter associated with the effective carrying capacity for species S_i. In this study, we assume that all parameters are independent of population density.

The system (1) incorporates a frequency-dependent Allee effect in the first equation, affecting species S_1 [4, 5]. This effect is captured by the critical compensation term $\left(\frac{S_1}{K_0} - 1\right)$, which directly influences the recruitment of individuals in species S_1. The parameter $K_0 > 0$ (with $0 < K_0 < K_1$) represents the minimum viable population size (MVPS), i.e., the smallest population size required for the persistence of species S_1, a concept widely studied in models incorporating the Allee effect [9, 10].

Furthermore, we assume that the birth rate exceeds the mortality rate for both populations,

$$\psi_1 > \delta_1 \quad \text{and} \quad \psi_2 > \delta_2, \tag{3}$$

and that population S_1 exhibits a higher survival capacity than S_2, i.e.,

$$\psi_2 < \psi_1, \quad \delta_2 > \delta_1, \quad \text{and} \quad r_2 < r_1. \tag{4}$$

The impulsive subsystem (2) describes the release of individuals from species S_2, where τ represents the release period and $u_n \in U$ denotes the number of individuals introduced at time $t = n\tau$. The control variable u_n is constrained by the available quantity of S_2, so that the set of admissible releases is $U = [0, u_{\max}]$, with $u_{\max} \geq 0$ denoting the maximum number of S_2 individuals that can be released at a given time.

The population of S_1 immediately after the n-th release, at $t = n\tau^+$, is given by $S_1(t^+) = \lim_{a \to 0^+} S_1(t + a)$.

The results concerning the existence, uniqueness, positivity, and uniform boundedness of the model solutions are presented in [1].

We now formulate an optimal control problem in which the model (1)-(2) serves as the control system. The goal is to minimize the number of released individuals of species S_2 while driving the population of S_1 below its survival threshold K_0 (associated with the Allee effect).

Mathematically, we seek to determine the sequence of optimal controls $u^* = (u_n^*)_{n=1}^N$, $u_n^* \in U$, that minimizes the cost functional.

$$J(u) = C_1(S_1(T) - (K_0 - \epsilon)) + C_2 \sum_{n=1}^{N} u_n, \tag{5}$$

subject to the model (1)-(2). We denote this optimal control problem as $\mathcal{P}$.

The term $S_1(T) - (K_0 - \epsilon)$ in the objective functional enforces that the final population of S_1 remains below K_0, its survival threshold, allowing a small tolerance $\epsilon > 0$. The constants C_1 and C_2 represent the costs associated with the control strategy, while $\sum_{n=1}^{N} u_n$ denotes the total release effort to be minimized. This problem is formulated over a finite time horizon $[0, T]$, where T defines the final intervention time. During this period, up to N releases can be performed, each belonging to the admissible set U.

To ensure the existence of solutions for this optimal control problem, we must show that there exists at least one element in U that satisfies (5) subject to the dynamical system (1)-(2). Therefore, we establish the following proposition.

Proposition 2.1. *There exists at least one n^* such that $u_n^* \in U$ satisfies $J(u)$, subject to* (1)-(2).

Proof. Since the set U is compact and the functional $J(u)$ is continuous (being a sum of continuous functions), the Weierstrass theorem ensures the existence of a control $u_n^* \in U$ that minimizes $J(u)$. □

Furthermore, having established the existence of at least one element in U that satisfies the optimal control problem, and considering that system (1)-(2) admits positive, unique, and uniformly bounded solutions (for each initial condition), and that the cost functional is continuous in u, we can apply Theorem (5.1) from Section III of [3]. Consequently, the existence of an optimal control $u^* = (u_n^*)_{n=1}^N$, with $u_n^* \in U$, for problem $\mathcal{P}$ is guaranteed.

Having ensured the existence of an optimal solution, the next section presents numerical results analyzing the influence of the parameters K_1 and K_2, which denote the effective carrying capacities for species S_1 and S_2, respectively.

3 Numerical Results

In this section, we present the numerical simulations of problem ($\mathcal{P}$), illustrating the dynamics of species S_1 and S_2 and the evolution of the optimal control for different values of the parameters K_1 and K_2, as detailed in Table 2. These parameters represent the effective carrying capacities of each species and directly influence their competitive interaction. The purpose of these simulations is to investigate how different parameter configurations affect the optimal release strategy of individuals from S_2 to achieve the replacement of S_1.

For these simulations, we consider a fixed release period of $\tau = 7$ days, a final intervention time of $T = 360$ days, and a control cost weighting factor of $C_1 = 1$ and $C_2 = \frac{1}{2}$. The parameter values used are presented in Table 1 ([4]). The optimal control problem was solved using the Gekko package, a Python library for dynamic optimization, optimal control, and nonlinear programming [2].

With this approach, we analyze the sensitivity of the problem with respect to K_1 and K_2, examining how variations in the effective carrying capacities influence the efficiency of the release strategies and the feasibility of replacing S_1 by S_2.

The results of the numerical simulations presented in Table 2 show the relationship between the parameters K_1 and K_2, the final intervention time $T = 360$ days, and the obtained values for the

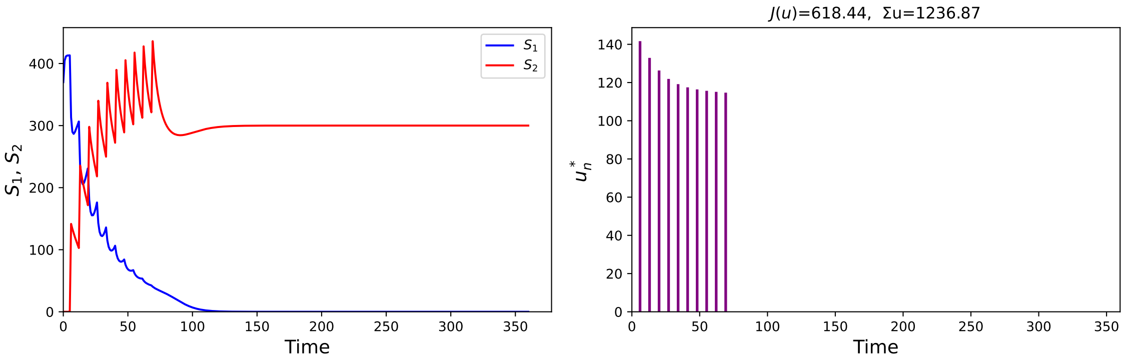
Table 1: Parameters for the model (1)-(2)

Parameter	Value	Range	Units	Description
ψ_1	0.32667	0.28 - 0.38	day^{-1}	Birth rate of species S_1
ψ_2	0.21333	0.18 - 0.25	day^{-1}	Birth rate of species S_2
δ_1	0.03333	1/8 - 1/42	day^{-1}	Death rate of species S_1
δ_2	0.06666	2/8 - 2/42	day^{-1}	Death rate of species S_2
K_0	30, 60, 60	-	-	Threshold S_1 population for species interaction

total release sum $\sum u_n$ and the minimum cost min $J(u)$. When K_1 increases from 374 to 674 (with $K_2 = 400$), the total number of released individuals ($\sum u_n = 4168.64$) rises significantly, reflecting the challenge of replacing a more abundant resident population. Conversely, when increasing K_2 from 400 to 674 (with $K_1 = 674$), $\sum u_n$ decreases to 1152.32, and the minimum cost drops to 576.16. This contrast demonstrates that a higher effective carrying capacity for S_2 facilitates its establishment, thereby reducing the required intervention effort. Hence, the parameters K_1 and K_2 together with the initial dominance of S_1 directly influence population dynamics and control efficiency.

Table 2: Numerical results for the optimization of problem $\mathcal{P}$.

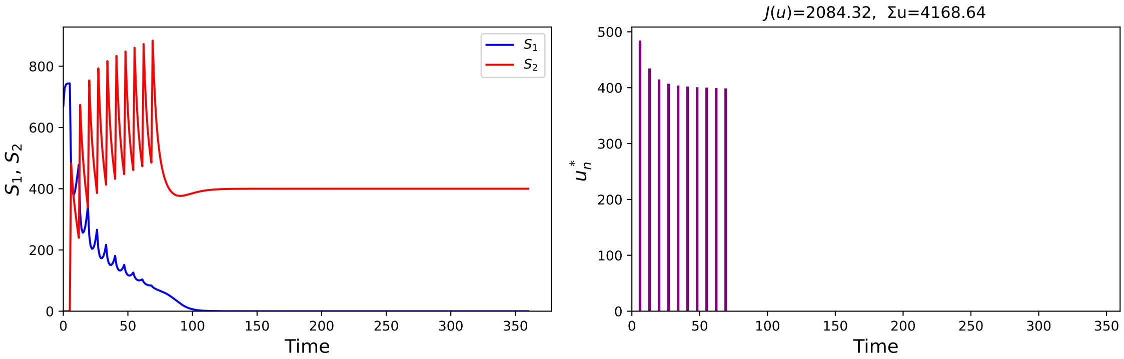
K_1, K_2, K_0	$T = 360$	
	$\sum u_n$	min $J(u)$
374, 300, 30	1236.87	618.44
674, 400, 60	4168.64	2084.32
674, 674, 60	1152.32	576.16



(a) (b)

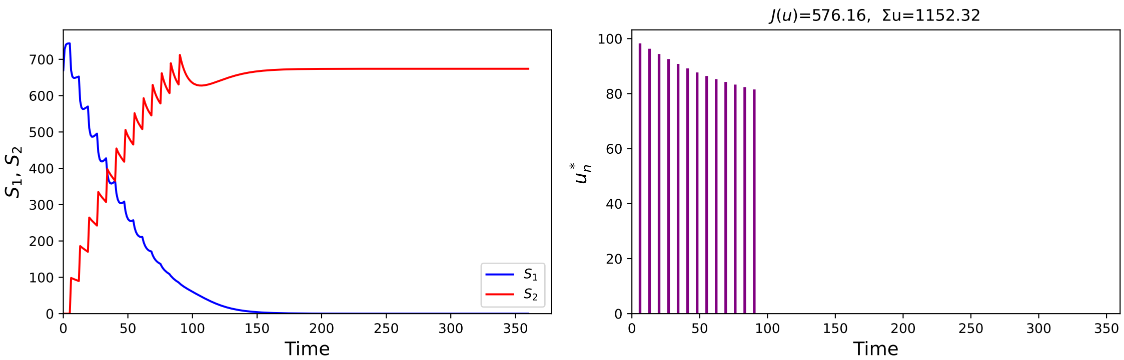
Figure 1: (a) Trajectory of S_1 and S_2 under optimal control for $K_1 = 374$ and $K_2 = 300$, with initial condition $(K_1, 0)$. (b) Optimal control u^* associated with the simulation in (a). Source: Figures by the authors.

Figures 1-3, associated with the results in Table 2, illustrate the optimal solutions for each combination of K_1 and K_2 in the population dynamics of S_1 and S_2 over time, along with the corresponding optimal impulsive controls u_n^*. All simulations consider an initial condition in which the system is entirely dominated by S_1, with no individuals of S_2 present. This initial configuration is crucial, as the density of S_1 determines the release intensity required for S_2 to establish itself competitively. The optimal strategy prioritizes a rapid reduction of S_1 while ensuring a stable establishment of S_2, with more intensive releases at the beginning of the intervention followed by a progressive decrease over time, an approach that minimizes the total cost while achieving the desired outcome.



(c) (d)

Figure 2: (c) Trajectory of S_1 and S_2 under optimal control for $K_1 = 674$ and $K_2 = 400$, with initial condition $(K_1, 0)$. (d) Optimal control u^* associated with the simulation in (c). Source: Figures by the authors.



(e) (f)

Figure 3: (e) Trajectory of S_1 and S_2 under optimal control for $K_1 = 674$ and $K_2 = 674$, with initial condition $(K_1, 0)$. (f) Optimal control u^* associated with the simulation in (e). Source: Figures by the authors.

These findings demonstrate the sensitivity of the optimal strategy to the model parameters, particularly K_1 and K_2, highlighting that the success and cost of population replacement depend not only on the initial density of S_1 but also on the effective carrying capacity available to S_2. The proposed approach enables a precise quantification of these effects and provides a valuable framework for supporting decision-making in biological intervention programs.

4 Final Considerations

Our results demonstrate that the effective carrying capacities (K_1, K_2) directly determine the intervention effort required for population replacement. Higher K_1 values demand significantly larger releases of S_2 to overcome the competitive advantage of S_1, whereas increased K_2 facilitates the establishment of S_2 with reduced effort.

The optimal strategy adapts to the system parameters by scheduling releases to maximize their impact during the most critical phases of S_1 suppression. These results provide valuable insights for designing intervention plans that support the development of efficient biological control strategies.

Furthermore, the proposed method offers an effective approach to minimizing the total intervention cost within an impulsive control framework. Unlike most studies that assume continuous release strategies, the use of impulsive control provides a more realistic and operationally feasible perspective for practical implementation.

Future studies could investigate the model's sensitivity to other system parameters, such as species mortality rates and their impact on control effectiveness. Additionally, incorporating a parameter that accounts for non-natural mortality—such as deaths caused by external factors like insecticide use or adverse environmental conditions—would enable a more comprehensive and realistic assessment of the optimal control problem. Such extensions would contribute to the development of more robust and effective strategies for replacing species S_1 with S_2.

Acknowledgments

J. C. S. A. would like to thank CAPES - Finance Code 001.

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