Proceeding Series of the Brazilian Society of Computational and Applied Mathematics, v. 12, n. 1, 2026.

Trabalho apresentado no XLIV CNMAC, Fundação Getulio Vargas - Rio de Janeiro - RJ, 2025.

Proceeding Series of the Brazilian Society of Computational and Applied Mathematics

Interval Contingent Derivative and Single-Level Interval Derivative

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Abstract. In this paper, the concept of the Interval Contingent Derivative is presented, which allows identifying a relationship between the contingent derivative of a compact convex-valued multifunction and the single-level derivative of an interval-valued function.

Keywords. Interval-valued functions, Single-level derivative, Multifunction, Contingent derivative of a multifunction

1 Introduction

Aubin and Frankowska, in [1], point out that in the second half of the twentieth century, the Polish mathematician Kuratowski gave the concept of multifunctions its proper significance (see [6]), while the renowned Bourbaki group focused solely on the study of functions as they are understood today, treating multifunctions as a particular case of the latter.

Although the concept of multifunctions was relegated to the background for many years, its wide range of applications led several mathematicians in the 1980s, including Aubin and Frankowska, to recognize the need to revisit the study of multifunctions.

More recently, in 2022, Leal et al. [7] introduced the concept of the single-level derivative for interval-valued functions, an idea developed based on Single-Level Constraint Interval Arithmetic (SLCIA), originally proposed in 2014 by Chalco, Lodwick and Bede [2].

In this context, observing an apparent similarity between an interval-valued function and a compact convex-valued multifunction, the objective of this study is to establish a relationship between the single-level derivative of an interval-valued function and the contingent derivative of a multifunction.

This paper is organized as follows: Section 2 defines the contingent derivative of multifunctions. Section 3 presents the single-level derivative for interval-valued functions. In Section 4, the concept of the interval contingent derivative for compact convex-valued multifunctions is introduced and compared with the single-level derivative for interval-valued functions. Finally, concluding remarks are provided.

2 Contingent Derivative of Multifunctions

Definition 2.1. *[3] Let X and Y be topological spaces. F is a multifunction from X to Y if to each $x \in X$ corresponds a subset $F(x)$ of Y.*

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A multifunction F from X to Y will be denoted by $F : X \rightsquigarrow Y$.

Definition 2.2. *[1] Let X and Y be metric spaces. A graph of a multifunction F from X to Y is denoted by $Graph(F)$ and is defined as:*

$$Graph(F) = \{(x, y) \in X \times Y \mid y \in F(x)\} . \tag{1}$$

A multifunction $F : X \rightsquigarrow Y$ is characterized by its graph.

Remark 2.1. *A function $f : X \to Y$ can be considered as a special multifunction $F : X \rightsquigarrow Y$ if $F(x) = \{f(x)\}$, i.e., if to each $x \in X$ corresponds to a singleton.*

Example 2.1. *Let $F : \mathbb{R} \rightsquigarrow \mathbb{R}$ be a multifunction defined by:*

$$F(x) = [1, 2]|x| = \begin{cases} [x, 2x], & \text{if } x \geq 0, \\ [-x, -2x], & \text{if } x < 0. \end{cases}$$

Thus,

$$Graph(F) = \left\{(x, y) \in \mathbb{R}^2 \,/\, y \in [1, 2]|x|\right\} .$$

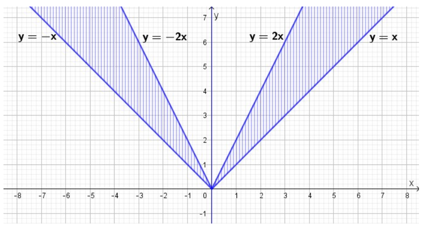

Figure 1: $Graph(F)$. Source: Authors

2.1 Contingent Derivative

In this section, we present some results on the analysis of multifunctions, considering an interval-valued function as a multifunction.

Definition 2.3. *[4] Let $(X, \| \, . \, \|)$ be a real normed space, $S \subset X, S \neq \emptyset$ and be $\overline{x} \in \overline{S}$. We call S the contingent cone in $\overline{x}$ to the set of vectors $h \in X$, such that there exist sequences $\{\lambda_n\} \subset \mathbb{R}^+$ and $\{x_n\} \subset S$ that satisfy:*

$$\lim_{n\to\infty} x_n = \overline{x} \;\; \textit{and} \;\; \lim_{n\to\infty} \lambda_n \left(x_n - \overline{x}\right) = h .$$

We will denote the cone contingent to S in $\overline{x}$ by $T_S\left(\overline{x}\right)$, then the above can be expressed symbolically as:

$$T_S(\overline{x}) = \{h \in X \mid \exists\{\lambda_n\} \subset \mathbb{R}^+, \exists\{x_n\} \subset S, x_n \to \overline{x} \textit{ and } \lambda_n \left(x_n - \overline{x}\right) \to h\} .$$

The vector $h \in T_S(\overline{x})$ is called the tangent vector to S at $\overline{x}$.

Remark 2.2. *According to Khan et al. [5], the contingent cone is also called the Bouligand tangent cone, or simply the tangent cone.*

Next, we present the contingent derivative for a multifunction.

DOI: 10.5540/03.2026.012.01.0297

Definition 2.4. *[1] Let X, Y be normed spaces, and let $F : X \rightsquigarrow Y$ be a multifunction. The contingent derivative of F on $(x, y) \in Graph(F)$ is a multifunction from X to Y, denoted by $DF(x, y)$, is defined as:*

$$Graph(DF(x,y)) = T_{Graph(F)}(x,y)\,. \tag{2}$$

Example 2.2. *Consider the example 2.1, it follows that:*

1. $Graph(DF(0,0)) = T_{Graph(F)}(0,0) = Graph(F)$.
2. $Graph(DF(2,2)) = T_{Graph(F)}(2,2) = \{(x,y) \in \mathbb{R}^2 \,/\, y \geq x\}\,.$
3. $Graph(DF(2,3)) = T_{Graph(F)}(2,3) = \mathbb{R}^2\,.$
4. $Graph(DF(2,4)) = T_{Graph(F)}(2,4) = \{(x,y) \in \mathbb{R}^2 \,/\, y \leq 2x\}\,.$

3 Single-Level Derivative of an Interval-Valued Function

3.1 Simple-Level Constrained Interval Arithmetic (SLCIA)

In 2014, Chalco, Lodwick and Bede [2] proposed the simple-level constrained interval arithmetic (SLCIA) as a variant of the constrained interval arithmetic (CIA) developed by Lodwick, see [8]. In these arithmetic properties, we can observe that the use of a single parameter λ is proposed for the constrained parametric representation involved in an operation, and it is also important to mention that the use of this arithmetic is appropriate when the values within the intervals vary in the same way.

Definition 3.1. *[2] Each interval $A = [\underline{a}, \overline{a}] \in \mathbb{I}$, where $\mathbb{I}$ is the interval space, is associated with a continuous function $A : [0,1] \to \mathbb{R}$, called the constrained function associated with A, such that:*

$$A = \left[\min_{0\leq\lambda\leq 1} A(\lambda), \max_{0\leq\lambda\leq 1} A(\lambda)\right]. \tag{3}$$

Definition 3.2. *[2] Given an interval $A = [\underline{a}, \overline{a}]$:*

1. *The increasing convex constraint function $A : [0,1] \to \mathbb{R}$ associated with the interval A is defined as:*
$$A(\lambda) = (1-\lambda)\underline{a} + \lambda\overline{a} = \underline{a} + \lambda\left(\overline{a} - \underline{a}\right), 0 \leq \lambda \leq 1. \tag{4}$$
2. *The decreasing convex constraint function $A : [0,1] \to \mathbb{R}$ associated with the interval A is defined as:*
$$A(\lambda) = \lambda\underline{a} + (1-\lambda)\overline{a} = \left(\underline{a} - \overline{a}\right)\lambda + \overline{a}, \;\; 0 \leq \lambda \leq 1. \tag{5}$$

Remark 3.1. *From now on, each interval will be associated with its increasing convex constraint function, which will be referred to as just constraint function.*

3.2 SL-Operations

Given the intervals $A = [\underline{a}, \overline{a}]$ and $B = \left[\underline{b}, \overline{b}\right]$ with constraint functions $A(\lambda)$ and $B(\lambda)$, respectively, and given a scalar α, the constrained arithmetic operations on single levels denoted as SL-operations are defined as follows:

1. SL-Multiplication of an interval A and a scalar α is defined as:

$$\alpha \odot A = \left[\min_{0\leq\lambda\leq 1} (\alpha \cdot A(\lambda)), \max_{0\leq\lambda\leq 1} (\alpha \cdot A(\lambda))\right] = [\min\{\alpha \cdot \underline{a}, \alpha \cdot \overline{a}\}, \max\{\alpha \cdot \underline{a}, \alpha \cdot \overline{a}\}]. \quad (6)$$

2. The SL-Operation $A * B$ is defined as:

$$A * B = \left[\min_{0\leq\lambda\leq 1} (A(\lambda) * B(\lambda)), \max_{0\leq\lambda\leq 1} (A(\lambda) * B(\lambda))\right], \quad (7)$$

where $* \in \{\oplus, \ominus, \otimes, \oslash\}$.

Remark 3.2. *Let $A = [\underline{a}, \overline{a}]$ and $B = [\underline{b}, \overline{b}]$ be intervals, then:*

1. $A \oplus B = \left[\underline{a} + \underline{b}, \overline{a} + \overline{b}\right]$.
2. $A \ominus B = \left[\min\{\underline{a} - \underline{b}, \overline{a} - \overline{b}\}, \max\{\underline{a} - \underline{b}, \overline{a} - \overline{b}\}\right]$.
3. *$A \oslash B$ exists whenever $0 \notin B$.*

3.3 SL-Derivative

Now, we review the concept of the single-level derivative of an interval-valued function, proposed by Leal et al. in [7], defined on the basis of single-level constrained interval arithmetic (SLCIA). Some theorems and an example are also presented.

Definition 3.3. *[9] A function of the form $F : [a, b] \to \mathbb{I}$ defined by $F(x) = \left[\underline{f}(x), \overline{f}(x)\right] \in \mathbb{I}$, such that $\underline{f}(x) \leq \overline{f}(x), \forall x \in [a, b]$ is called an interval-valued function.*

Definition 3.4. *[7] Let $F : [a, b] \to \mathbb{I}$ be an interval-valued function, $x_0 \in]a, b[$ and $h \neq 0$ a real number such that $x_0 + h \in]a, b[$, then the single-level derivative of F at x_0, which we denote by $D_{SL}F(x_0)$, is defined as $D_{SL}F(x_0)$:*

$$D_{SL}F(x_0) = \lim_{h\to 0} \frac{1}{h} \odot (F(x_0 + h) \ominus F(x_0)). \quad (8)$$

If the above limit exists, F is said to be single-level differentiable (SL-differentiable, as short) at x_0.

Theorem 3.1. *[7] Let $F : T \to \mathbb{I}$ be an interval-valued function defined by $F(x) = \left[\underline{f}(x), \overline{f}(x)\right] \in \mathbb{I}$, where T is an open or closed interval. If $\underline{f}$ and $\overline{f}$ are differentiable functions at $x_0 \in T$, then F is SL-differentiable at x_0 such that:*

$$D_{SL}F(x_0) = \left[\min_{0\leq\lambda\leq 1} \left\{(1-\lambda)\underline{f}'(x_0) + \lambda\overline{f}'(x_0)\right\}, \max_{0\leq\lambda\leq 1} \left\{(1-\lambda)\underline{f}'(x_0) + \lambda\overline{f}'(x_0)\right\}\right]. \quad (9)$$

Theorem 3.2. *Let $F : T \to \mathbb{I}$ be an interval-valued function defined by $F(x) = \left[\underline{f}(x), \overline{f}(x)\right] \in \mathbb{I}$. If $\underline{f}$ and $\overline{f}$ are functions whose lateral derivatives exist in $x_0 \in T$, then:*

i) F is SL-differentiable on the right at x_0 and

$$D_{SL}F_+(x_0) = \left[\min_{0\leq\lambda\leq 1} \left\{(1-\lambda)\underline{f}'_+(x_0) + \lambda\overline{f}'_+(x_0)\right\}, \max_{0\leq\lambda\leq 1} \left\{(1-\lambda)\underline{f}'_+(x_0) + \lambda\overline{f}'_+(x_0)\right\}\right]. \quad (10)$$

ii) F is SL-differentiable on the left at x_0 and

$$D_{SL}F_-(x_0) = \left[\min_{0\leq\lambda\leq 1}\left\{(1-\lambda)\underline{f}'_-(x_0)+\lambda\overline{f}'_-(x_0)\right\}, \max_{0\leq\lambda\leq 1}\left\{(1-\lambda)\underline{f}'_-(x_0)+\lambda\overline{f}'_-(x_0)\right\}\right]. \quad (11)$$

Theorem 3.3. *Let $F : T \to \mathbb{I}$ be an interval-valued function defined by $F(x) = \left[\underline{f}(x), \overline{f}(x)\right] \in \mathbb{I}$, then F is SL-differentiable in $x_0 \in T$ if and only if the right and left single-level derivatives of F in x_0 exist, and are equal, i.e. $D_{SL}F_+(x_0) = D_{SL}F_-(x_0)$.*

Example 3.1. *Given the interval-valued function $F : \mathbb{R} \to \mathbb{I}$ defined by $F(x) = [|x|, 2|x|]$, it follows that:*

a) If $x > 0$, $F(x) = [x, 2x]$, and since $\underline{f}(x) = x$ and $\overline{f}(x) = 2x$ are differentiable functions at $x > 0$, by Theorem 3.1 it follows that:

$$D_{SL}F(x) = \left[\min_{0\leq\lambda\leq 1}\{\lambda+1\}, \max_{0\leq\lambda\leq 1}\{\lambda+1\}\right] = [1, 2].$$

b) Similarly, if $x < 0$, $F(x) = [-x, -2x]$, then $\forall\ x < 0$ it follows that:

$$D_{SL}F(x) = \left[\min_{0\leq\lambda\leq 1}\{-\lambda-1\}, \max_{0\leq\lambda\leq 1}\{-\lambda-1\}\right] = [-2, -1].$$

c) The interval-valued function $F(x) = [|x|, 2|x|]$ is not SL-differentiable at $x = 0$. Since:

i) $D_{SL}F_+(0) = \left[\min_{0\leq\lambda\leq 1} F'_+(\lambda)(0), \max_{0\leq\lambda\leq 1} F'_+(\lambda)(0)\right] = \left[\min_{0\leq\lambda\leq 1}\{\lambda+1\}, \max_{0\leq\lambda\leq 1}\{\lambda+1\}\right] = [1, 2].$

ii) $D_{SL}F_-(0) = \left[\min_{0\leq\lambda\leq 1} F'_-(\lambda)(0), \max_{0\leq\lambda\leq 1} F'_-(\lambda)(0)\right] = \left[\min_{0\leq\lambda\leq 1}\{-\lambda-1\}, \max_{0\leq\lambda\leq 1}\{-\lambda-1\}\right] = [-2, -1].$

Therefore, according to Theorem 3.3 F is not SL-differentiable in 0.

4 Interval Contingent Derivative

From the Examples 2.2 and 3.1 it can be seen that the contingent derivative of a multifunction, in general, results in a non-compact set, while determining the single-level derivative of an interval-valued function results in an interval; therefore, there is no direct relationship between the two derivatives. However, in order to establish more clearly a relationship between these derivatives, the concept of the Interval Contingent Derivative is proposed below.

Definition 4.1. *Let X, Y be normed spaces, and $F : X \rightsquigarrow Y$ be a multifunction given by $F(x) = [\underline{f}(x), \overline{f}(x)]$. The interval contingent derivative of F at $x \in Dom(F)$ is denoted and defined by:*

$$Graph(\boldsymbol{D}F(x)) = \bigcap_{y\in\left[\underline{f}(x),\overline{f}(x)\right]} T_{Graph(F)}(x, y). \quad (12)$$

Example 4.1. *Considering the multifunction of the Example 2.1, then:*

$$Graph(\boldsymbol{D}F(1)) = \bigcap_{y\in[1,2]} T_{Graph(F)}(1, y),$$

whose graph is:

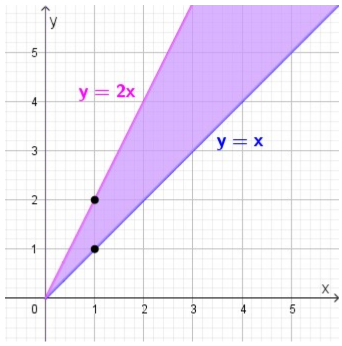

Figure 2: $Graph(\mathbf{D}F(1))$. Source: Authors

From Example 3.1 it follows that:

$$D_{SL}F(x) = \begin{cases} [1,2], & \text{if } x > 0, \\ [-2,-1], & \text{if } x < 0\,. \end{cases}$$

Then, $D_{SL}F(1) = [1,2]$. It can be seen that the graph of the interval contingent derivative at $x = 1$ is the set of points of $\mathbb{R}^2$ bounded by the straight lines $y = x$ and $y = 2x$, $\forall\ x > 0$, whose slopes vary in the interval $[1,2]$, which coincides with the result provided by the single derivative at that point, i.e., $D_{SL}F(1) = [1,2]$.

5 Final Considerations

Based on the findings presented in this article, there is no direct relationship between the contingent derivative of a compact convex-valued multifunction and the single-level derivative of the same multifunction when considered as an interval-valued function. However, through the concept of the interval contingent derivative, it is observed that, at a given point, its graph consists of a set of points in $\mathbb{R}^2$ bounded by straight lines whose slopes vary within a range of values that coincide with the result provided by the single-level derivative of the multifunction considered as an interval-valued function.

References

[1] J. P. Aubin and H. Frankowska. **Set-Valued Analysis**. Birkhäuser Boston, 2009. DOI: 10.1007/978-0-8176-4848-0. URL: https://doi.org/10.1007%2F978-0-8176-4848-0.

[2] Y. Chalco-Cano, W. A. Lodwick, and B. Bede. "Single level constraint interval arithmetic". In: **Fuzzy Sets and Systems** 257 (Dec. 2014), pp. 146–168. DOI: 10.1016/j.fss.2014.06.017. URL: https://doi.org/10.1016%2Fj.fss.2014.06.017.

[3] A. Geletu. "Introduction to topological spaces and set-valued maps (Lecture notes)". In: **Ilmenau, Germany: Institute of Mathematics, Department of Operations Research and Stochastics, Ilmenau University of Technology** (2006).

[4] J. Jahn. **Introduction to the theory of nonlinear optimization**. Springer Nature, 2020.

[5] A. A. Khan, C. Tammer, and C. Zalinescu. **Set-valued optimization**. Springer, 2016.

[6] K. Kuratowski. "Introduction". In: **Topology**. Ed. by K. Kuratowski. Academic Press, 1966, pp. 1–37. ISBN: 978-0-12-429201-7. DOI: https://doi.org/10.1016/B978-0-12-429201-7.50005-3. URL: https://www.sciencedirect.com/science/article/pii/B9780124292017500053.

[7] U. A. S. Leal, G. Maqui, G. N. Silva, and W. A. Lodwick. "Single-Level Differentiability for Interval-valued Functions". In: **Proceeding Series of the Brazilian Society of Computational and Applied Mathematics**. SBMAC, Dec. 2022. DOI: 10.5540/03.2022.009.01.0304. URL: https://doi.org/10.5540%2F03.2022.009.01.0304.

[8] W. A. Lodwick. "Constrained interval arithmetic (Technical Report 138). Denver, CO: University of Colorado at Denver". In: **Center for Computational Mathematics** (1999).

[9] L. Stefanini, M. L. Guerra, and B. Amicizia. "Interval Analysis and Calculus for Interval-Valued Functions of a Single Variable. Part I: Partial Orders, gH-Derivative, Monotonicity". In: **Axioms** 8.4 (Oct. 2019), p. 113. DOI: 10.3390/axioms8040113. URL: https://doi.org/10.3390%2Faxioms8040113.

