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Time-Discretization Schemes in Solving a Heat Diffusion Model: FVM Implementation

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Differential equations serve as a powerful tool for modeling physical phenomena in which a key quantity varies concerning one or more other factors. For instance, in renewable energy systems like wind turbines, differential equations are used to model fluid dynamics to predict wind flow patterns and optimize turbine designs for maximum energy extraction [1]. Similarly, in solar energy, differential equations are used to model heat transfer processes within thermal energy storage systems, such as phase change materials or sensible heat storage systems. These equations describe the transient behavior of temperature distribution within the storage medium as heat is absorbed or released during charging and discharging cycles. [3]. These problems are tackled by solving momentum or heat transport equations, respectively. These equations, prominent in computational fluid dynamics, provide a foundational framework for analyzing such problems [2].

Numerical methods are mostly used to solve any differential equation. These methods replace existing derivatives with algebraic expressions involving the unknown function and deliver a solution for a discrete number of points with a certain error. The greater this number of points, it is expected that the numerical solution will be closer to the exact solution [4]. This work proposes to analyze time-discretization schemes to solve a heat diffusion model. The latter is represented by a two-dimensional unsteady-state heat diffusion equation, given by

$$\frac{\partial(\rho T)}{\partial t} = \frac{\partial}{\partial x}\left(\frac{k}{C_p}\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{k}{C_p}\frac{\partial T}{\partial y}\right), \tag{1}$$

where ρ, k, and C_p are the density, thermal conductivity, and isobaric specific heat of the fluid, respectively.

The differential equation is solved using the finite volume method (FVM). This method is selected because of the physical interpretation of the equations. The fully discretized form for the equation (1), considered for this work, is given by

$$\begin{aligned}\frac{\rho C_p(T_p - T_p^\circ)}{\Delta t} = \theta\left[\frac{k\Delta y}{\Delta x}(T_e + T_w) + \frac{k\Delta x}{\Delta y}(T_n + T_s)\right] + \\ +(1-\theta)\left[\frac{k\Delta y}{\Delta x}(T_e^\circ + T_w^\circ) + +\frac{k\Delta x}{\Delta y}(T_n^\circ + T_s^\circ)\right],\end{aligned} \tag{2}$$

where firstly, the temporal term of (1) is discretized using a first-order (backward) differencing scheme. To discretize the temporal terms on the right side of Equation (1), we introduce a weighting parameter, denoted as θ, with values ranging between 0 and 1. The schemes considered were the

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explicit method ($\theta = 0$), the Crank-Nicholson method ($\theta = 0.5$), and the fully implicit method ($\theta = 1$). The explicit and fully implicit have a first-order error convergence. The Crank-Nicholson method has a second-order error convergence. The central difference method is employed for spatial discretization to estimate the temperature distribution at any cell P. This method has a second-order error convergence [5].

To verify the code implementation, the following manufactured solution was used:

$$f(x, y, t) = \frac{4}{\pi^2} e^{-2\pi^2 \alpha t} \sin(\pi x) \sin(\pi y). \tag{3}$$

The numerical verification was done considering the boundary values of the solution in (3), and they were input into the code implementation. Afterward, the numerical solution was compared with the exact solution at the final instant. For this, the maximum error was evaluated in all cells as the number of cells increased. The overall numerical method converged as first-order accurate, as expected.

A classic heat diffusion problem was solved with an initial temperature and prescribed temperature at the boundaries to assess the aforementioned temporal schemes. The conclusion was that the explicit and Crank-Nicholson methods have a maximum size for the time step. Therefore, obtaining any temperature distribution behavior over time can take a significant amount of time. On the other hand, the fully implicit method does not have the time step size limitation, but it is first-order accurate. This work is an initial step in investigating higher-order schemes to discretize the temporal term of equation (1) since it will allow for a more accurate solution for more complex problems.

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