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Quantum Compact Graphs

Thiago Assis[1] Pedro Baptista[2] Gabriel Coutinho[3]
Departamento de Ciência da Computação, Universidade Federal de Minas Gerais, Belo Horizonte, MG
Chris Godsil[4]
Dept. of Combinatorics and Optimization, University of Waterloo, Waterloo, ON, Canada

One of the main topics of research on graphs is its automorphism group $\mathrm{Aut}(G)$. Although it is known that most graph have trivial automorphism group [2], it is not yet known if the problem of deciding for a given graph G if it has trivial $\mathrm{Aut}(G)$ is in P or is it NP-complete. Elements of $\mathrm{Aut}(G)$ can be viewed as permutations matrices P such that

$$PA(G) = A(G)P, \tag{1}$$

where $A(G)$ (we write A for short) is the adjacency matrix of the graph, i.e, a 01 symmetric matrix with 1 at entry ij if and only if i and j are adjacent. One way of relaxing this definition is to allow P to be a doubly stochastic matrix, i.e., the entries of each row and column form a probability distribution such that (1) holds. We define $S(A)$ to be the set of all such matrices.

The set $S(A)$ is bounded and convex, therefore it is a polytope. Also, all automorphisms of G are in $S(A)$. Furthermore, it can be shown that they must be vertices of $S(A)$.

One of the consequences of all of these points is that one can find all automorphisms of the graph by visiting each vertex of the set $S(A)$ through standard linear programming techniques. One advantage of this is that we do not need to verify all permutations, as only automorphisms of G will be there. But there might be far more vertices of $S(A)$ which are not permutations.

This is one of the motivations to study compact graphs. Compact graphs are graphs such that all extreme points of $S(A)$ are permutations. This in turn says that it is simpler to find the automorphisms of the graph. In [5, 6], it is shown that trees, cycles and disjoint unions of compact graphs are compact. Moreover, Godsil [3] shows that if a regular graph is compact, then it is generously transitive, i.e., there are transpositions in $\mathrm{Aut}(G)$ between any two vertices.

One relaxation of isomorphism is the concept of quantum isomorphism (see [1] for reference). This can be motivated through the idea of a game in which two players that are physically separated receive vertices of a graph, and they must reply with vertices from another graph (both are known prior to the game start). They must answer with equal vertices if they receive equal vertices, and adjacent/non-adjacent if they receive adjacent/non-adjacent vertices. They win the game if they answer correctly in a multi-round iteration of the game.

It can be shown that they win the game using classical strategies if and only if the graphs have an isomorphism. In the quantum version, both players have access to a quantum state and can perform measurements to decide which vertex to answer with. If the players win the game, it is said that the graphs are quantum isomorphic. Determining if two graphs are quantum isomorphic is undecidable, although there are examples of graphs which are not isomorphic, but are quantum isomorphic (both can be seen in [1]).

[1]thiago.assis@dcc.ufmg.br
[2]pedro.baptista@dcc.ufmg.br
[3]gabriel@dcc.ufmg.br
[4]cgodsil@uwaterloo.ca

Our work tries to connect the set $S(A)$ with this notion of quantum isomorphisms. The first point is that an automorphism is an isomorphism from the graph to itself, so what we said also applies for quantum automorphisms (see [4]).

Another point is that a quantum automorphism can be viewed as a quantum permutation, a generalization of permutation matrices. A quantum permutation is a matrix Q in which its entries are orthogonal projection matrices and entries of rows and columns must now sum to the identity of proper dimension. Note that we can map Q to P, a doubly stochastic matrix, by taking each entry of Q, that is the matrix in Q_{ij}, and making

$$P_{ij} = \mathrm{Tr}(Q_{ij}\rho), \tag{2}$$

where ρ is a positive semi-definite matrix with trace 1 and Tr is the trace of a matrix. If the dimension of the entries in Q is d, then

$$Q(A \otimes I_d) = (A \otimes I_d)Q, \tag{3}$$

where $\otimes$ is the tensor product, defines Q to be a quantum automorphism of the graph. Note that in this case, the matrix P from (2) is in $S(A)$. In this paper, we work on a novel generalization of compact graphs. We introduce the definition of a quantum compact graph. It is a graph in which all vertices from $S(A)$ come from a quantum automorphism of the graph, as in (2).

We show that although compact graphs are quantum compact, there are examples that are neither compact nor quantum compact. We show that all maps from the entries of Q to a doubly stochastic matrix are of the form we defined in (2). Furthermore, if we allow Q to have positive semidefinite entries instead, we prove that all matrices from $S(A)$ can be mapped as we proposed.

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