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Three-Term Recurrence Relation for Quasi-Orthogonal Polynomials

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Let us consider $P = \{P_n(x)\}_{n\geq 0}$ a sequence of orthogonal polynomials with respect to $w(x) = e^{-v(x)}$ on an interval $[a, b]$. We assume that v is real and differentiable in (a, b), and all moments of the weight exist. We will also consider $\{p_n(x) = \gamma_n x^n + \ldots + \gamma_1 x + \gamma_0\}$ be the sequence of orthonormal polynomials obtained from P.

The concept of orthogonal polynomials has been an important tool in the analysis of a large variety of problems in mathematics and engineering, like moment problems and numerical quadrature, for example. Afterward, many other concepts related to orthogonal polynomials were proposed, as quasi-orthogonal polynomials [3, 4, 7, 8], among others.

An important property in the theory of orthogonal polynomials is that the polynomials in the orthonormal sequence $\{p_n(x)\}_{n\geq 0}$ satisfy a three-term recurrence relation of the form

$$xp_n(x) = a_{n+1}p_{n+1}(x) + \alpha_n p_n(x) + a_n p_{n-1}(x),\ n \geqslant 1, \tag{1}$$

with initial conditions $p_0(x) = \gamma_0$, $p_1(x) = \gamma_1(x - \alpha_0)$, where

$$\begin{aligned} a_n &= \int_a^b x p_n(x) p_{n-1}(x) w(x) dx = \frac{\gamma_{n-1}}{\gamma_n}, n \geq 1, \\ \alpha_n &= \int_a^b x p_n^2(x) w(x) dx, n \geq 0. \end{aligned}$$

In this work, our focus is dealing with the quasi-orthogonal polynomials. So, let R_n be a polynomial of exact degree $n \geq r$. If R_n satisfies the conditions

$$\int_a^b x^k R_n(x) w(x) dx \begin{cases} = 0, & \text{for } k = 0, \ldots, n-1-r \\ \neq 0, & \text{for } k = n-r \end{cases}, \tag{2}$$

where w is a positive weight function on $[a, b]$, then R_n is quasi–orthogonal of order r on $[a, b]$ with respect to w.

In [3] we can find a more general definition of quasi-orthogonality, where the following result is considered:

Theorem 1. *Let $\{P_n(x)\}_{n\geq 0}$ be the sequence of orthogonal polynomials on $[a, b]$ with respect to a positive weight function $w(x)$. A necessary and sufficient condition for a polynomial R_n of degree n to be quasi-orthogonal of order r on $[a, b]$ with respect to w is that*

$$R_n(x) = c_{n,0}P_n(x) + c_{n,1}P_{n-1}(x) + \cdots + c_{n,r}P_{n-r}(x),$$

where $c_{n,i}$, $i = 0, \ldots, r$, are numbers which can depend on n and $c_{n,0}c_{n,r} \neq 0$.

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Without loss of generality, let us consider

$$R_n(x) = p_n(x) + c_{n,1}p_{n-1}(x) + \cdots + c_{n,r}p_{n-r}(x).$$

The concept of quasi-orthogonality seems to have been introduced by Riesz [7], for $r = 1$. Féjer [4] considered the case $r = 2$ and the general case was first studied by Shohat [8]. Furthermore, there are many problems involving linear combinations of orthogonal polynomials, as mentioned in [5], and many authors studied this topic as [1, 2], for example.

Chihara [3] shows that the quasi-orthogonal polynomials R_n satisfy a three-term recurrence relation of the form

$$A_{n+1}(x)R_{n+1} + B_{n+1}(x)R_n(x) + C_{n+1}(x)R_{n-1}(x) = 0, n \geq 1, \tag{3}$$

where $A_n(x)$, $B_n(x)$ and $C_n(x)$ are polynomials in x.

For $r = 1$, i.e., $R_n(x) = p_n(x)+c_{n,1}p_{n-1}(x)$, the authors in [6] show that the recurrence relation (3) is the form

$$a_{n+1}(x)R_{n+1}(x) = [r_n(x)l_{n-1}(x) - c_{n-1,1}a_nl_n(x)]R_n(x) - a_{n-1}l_n(x)R_{n-1}(x), \tag{4}$$

where $r_n(x) = x - \alpha_n + c_{n+1,1}a_{n+1}$ and $l_n(x) = c_{n,1}r_n(x) + a_n$. Furthermore, they derive difference and differential equations satisfied by $R_n(x)$ from this relation, while also analyzing the orthogonality of quasi-orthogonal polynomials.

Inspired by the investigations outlined in [6], we derive a three-term recurrence relation for the quasi-orthogonal polynomials when $r = 2$. We apply these polynomials in contexts involving differential equations and the orthogonality of $R_n(x)$.

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