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Optimal Vaccination Strategies on Networks and in Metropolitan Areas

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This study presents a mathematical model to determine optimal vaccination strategies in metropolitan areas and general networks of cities, considering commuting patterns. The epidemiological model utilizes a compartmental SIR (Susceptible-Infected-Recovered) framework, including a vaccination rate by city, which acts as a control function, and equal birth and death rates to ensure a constant population. Commuting patterns are incorporated through a weighted adjacency matrix P such that p_{ij} measures the proportion of individuals who live in city i and work in city j, and a parameter α that weighs day and night periods. During the day, commuting affects the infection dynamics since individuals from different cities interact [4]. The complete model [3] is

$$\begin{aligned}\frac{dS_i}{dt}(t) &= \mu - \alpha\beta_i S_i I_i - (1-\alpha)S_i \sum_{j=1}^{K} \beta_j p_{ij} I_j^{\text{eff}} - \mu S_i,\\ \frac{dI_i}{dt}(t) &= \alpha\beta_i S_i I_i + (1-\alpha)S_i \sum_{j=1}^{K} \beta_j p_{ij} I_j^{\text{eff}} - \gamma I_i - \mu I_i,\\ \frac{dR_i}{dt}(t) &= \gamma I_i - \mu R_i,\end{aligned} \tag{1}$$

where β is the infection rate, γ is the recovery rate and μ is the birth and death rate. From this, we can define the basic reproduction number $\mathcal{R}_0$.

Building upon this model, we formulate an optimal control problem that aims to minimize a balance between the number of hospitalizations and vaccinations

$$J[u_1, \ldots, u_K] := \sum_{i=1}^{K} c_v n_i \int_0^T u_i(t) S_i(t)\, dt + c_h r_h n_i \int_0^T I_i(t)\, dt, \tag{2}$$

where n_i is the population size of city i, r_h the hospitalization rate. This problem is subject to constraints imposed by a per-time unit vaccination application limit and a weekly availability cap [2]

$$u_i(t)S_i(t) \le v_i^{\max}, \quad \text{for } i = 1, \ldots, K, \text{ and } \sum_{i=1}^{K} n_i \int_0^t u_i(t) S_i(t)\, dt \le D(t), \tag{3}$$

both for $t \in [0, T]$, where $v_i^{\max}$ is a daily cap of vaccination and $D(t)$ represents the accumulated availability of vaccines from the central producer. These caps, derived from real-world scenarios, lead to theoretical mixed control-state and pure-state constraints. They pose theoretical and

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numerical challenges, adding layers of complexity to the model's implementation. From the theoretical aspect, there are a lot of open questions since the *Pontryagin Maximum Principle* leads to measure-type adjoint variables. Additionally, the control problem is affine, which has practical relevance in opposition to additional mathematical difficulties [1]. In particular, a closed-form expression for the optimal control as a function of the state and adjoint variables has not yet been identified. This contrasts, for example, with quadratic costs. We argue that this approach yields a more realistic model for formulating vaccination strategies. Another difficulty this analysis brings is the potential high dimensionality of analyzing a multi-city policy formulation. This complicates numerical simulations and real-world applications.

The key findings include tighter upper and lower bounds for the model's basic reproduction number, particularly in the case of a metropolitan area, and the analysis of the control problem. For the latter, we study the existence of solution and its *bang-bang* shape, providing a partial proof for $n = 1$, with directions for a generalized version of the theorem. Numerical experiments demonstrate how the bounds on the reproduction number provide insights into the disease dynamics. In particular, we showed that $\mathcal{R}_0$ is close to the basic reproduction number of the capital city, if it were isolated, and most influenced by its infection rate. Theoretical analysis and numerical simulations contribute to understanding the effectiveness of control measures in mitigating the disease dynamics, emphasizing the need for incorporating spatial heterogeneity. As a practical example, we considered commuting and population data from the Rio de Janeiro metropolitan area. By implementing the optimal control solution, we observed the positive impact of utilizing this approach against constant vaccination rates.

The research highlights the importance of prioritizing vaccination in the capital to control the disease spread more effectively, as we depicted in our numerical simulations. This model serves as a tool to improve resource allocation of vaccines in epidemic control across metropolitan regions. Additionally, it provides a basis for furthering our understanding of how strategic interventions can impact overall disease dynamics. Future research will refine this strategy and explore theoretical aspects of constrained control-affine problems using their inherent geometric characteristics.

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