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Sensitivity of fuzzy f-Xor implications

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ABSTRACT

Since the degrees of certainty of fuzzy connectives are only approximately defined, it is reasonable to study the robustness of corresponding operator based on the sensitive to small changes in the inputs of such logical operation [3, 6, 7]. As the main contribution, the study of the pointwise sensitive of f-X(N)or class [2] is introduced, restricted here to the fuzzy connective E_{T_P,S_P,N_S} and corresponding fuzzy f-Xor implication E_{T_P,S_P,N_S}. The endpoints of $U = [0,1]$ are considered faced on the antitonicity of f-X(N)or class.

A function $\mathsf{E}(\mathsf{D}) : U^2 \to U$ is a *fuzzy exclusive or* (Xor) (*fuzzy exclusive nor* (XNor)) if it satisfies the following properties, for all $x, y \in U$:

E0: $\mathsf{E}(1,1) = \mathsf{E}(0,0)$=0 and $\mathsf{E}(1,0)$=1; **D0:** $\mathsf{D}(1,1) = \mathsf{D}(0,0)$=1 and $\mathsf{D}(0,1)$=0;
E1: $\mathsf{E}(x,y) = \mathsf{E}(y,x)$; **D1:** $\mathsf{D}(x,y) = \mathsf{D}(y,x)$;
E2(i): If $x \le y$ then $\mathsf{E}(0,x) \le \mathsf{E}(0,y)$; **D2(i):** If $x \le y$ then $\mathsf{D}(0,x) \ge \mathsf{D}(0,y)$;
E2(ii): If $x \le y$ then $\mathsf{E}(1,x) \ge \mathsf{E}(1,y)$. **D2(ii):** If $x \le y$ then $\mathsf{D}(1,x) \le \mathsf{D}(y,1)$.

Let T, S and N be a t-norm, a t-conorm and a fuzzy negation, respectively. By [1, Prop. 3.4], the function $\mathsf{E}_{T,S,N}(\mathsf{D}_{S,T,N}) : U^2 \to U$ defined as

$$\mathsf{E}_{T,S,N}(x,y) = T(S(x,y), N(T(x,y))); \qquad (\mathsf{D}_{S,T,N}(x,y) = S(T(x,y), N(S(x,y)))). \tag{1}$$

is a fuzzy f-Xor connective. Moreover, by a function $I_{\mathsf{E}_{T,S,N},T,N}(J_{\mathsf{D}_{S,T,N},S,N}) : U^2 \to U$, called a fuzzy *$f$-Xor-(co)implication*, is given by

$$I_{\mathsf{E}_{T,S,N},T,N}(x,y) = \mathsf{E}_{T,S,N}(x, N(T(x,y))); \quad J_{D,S,N}(x,y) = \mathsf{D}_{S,T,N}(x, N(S(x,y))). \tag{2}$$

Proposition 1. *Consider $N_S : U \to U$, $N_S(x) = 1 - x$, $S_P, T_P : U^2 \to U$, $S(x,y) = x + y - xy$ and $T(x,y) = xy$. The fuzzy f-Xor E_{T_P,S_P,N_S} and related f-Xor implication $I_{\mathsf{E}_{T_P,S_P,N_S},T_P,N_S}$ are given as*

$$\mathsf{E}_{T_P,S_P,N_S}(x,y) = x + y - xy - x^2y - xy^2 + x^2y^2 \text{ and } I_{\mathsf{E}_{T_P,S_P,N_S},T_P,N_S}(x,y) = (1 - xy + x^2y)(1 - x + x^2y). \tag{3}$$

By [6, 8], the main results of a δ sensitivity of f at point $\mathbf{x}$ (or a pointwise sensitivity) on U, referred as $\Delta_f(\mathbf{x},\delta)$ and related to the fuzzy f-Xor connective is discussed. For that, based on [6, Theorem 1], consider $f : U^2 \to U$, $\delta \in U$ and $\mathbf{x} = (x,y) \in U^2$.

(i) If f is an monotone function, which means, $x \le x', y \le y' \Rightarrow f(x,y) \le f(x',y')$ then

$$\Delta_f(\mathbf{x},\delta) = [f(\mathbf{x}) - f((x-\delta) \vee 0, (y-\delta) \vee 0)] \vee [f((x+\delta) \wedge 1), (y+\delta) \wedge 1) - f(\mathbf{x})] \tag{4}$$

(ii) If $f : U \to U$ be a reverse order function, such that $x \le y \Rightarrow f(x) \ge f(y)$, then

$$\Delta_f(x,\delta) = [f(x) - f(x+\delta) \wedge 1)] \vee [f((x-\delta) \vee 0) - f(x)]. \tag{5}$$

(iii) If f verifies both properties, 1−place antitonicity and 2−place isotonicity, then it holds that

$$\Delta_f(\mathbf{x},\delta) = [f(\mathbf{x}) - f((x+\delta) \wedge 1, (y-\delta) \vee 0)] \vee [f((x-\delta) \vee 0), (y+\delta) \wedge 1) - f(\mathbf{x})] \tag{6}$$

Based on the monotonicity of S_P and T_P, it holds that:

$$\Delta_{S_P}((0,0),\delta) = 2\delta - \delta^2 = \Delta_{T_P}((1,1),\delta); \Delta_{S_P}((1,1),\delta) = \delta^2 = \Delta_{T_P}((0,0),\delta);$$
$$\Delta_{S_P}((0,1),\delta) = \delta = \Delta_{T_P}((1,0)\delta); \qquad \Delta_{S_P}((1,0),\delta) = \delta = \Delta_{T_P}((0,1),\delta).$$

The δ sensitivity of E_{S_P,T_P,N_S} at point $\mathbf{x}$ is presented in the following proposition.

Proposition 2. *Consider* $\Delta_{\mathsf{E}_{S_P,T_P,N_S}} : U^2 \to U$ *and* $\delta \in U$. *Then it holds that:*

$$\text{If} \quad \mathbf{x} = (0,0) \quad \text{then } \Delta_{\mathsf{E}_{S_P,T_P,N_S}}(\mathbf{x},\delta) = \Delta_{S_P}(\mathbf{x},\delta) - \Delta_{T_P}(\mathbf{x},\delta);$$
$$\text{If} \quad \mathbf{x} = (1,1) \quad \text{then } \Delta_{\mathsf{E}_{S_P,T_P,N_S}}(\mathbf{x},\delta) = \Delta_{S_P}(\mathbf{x},\delta) + \Delta_{T_P}(\mathbf{x},\delta).$$

Proposition 3. *Consider* $I_{\mathsf{E}_{T_P,S_P,N_S},T_P,N_S} : U^2 \to U$ *and* $\delta \in U$. *Then it holds that:*

$$\text{If} \quad \mathbf{x} = (0,0) \quad \text{then } \Delta_{I_{\mathsf{E}_{T_P,S_P,N_S},T_P,N_S}}(\mathbf{x},\delta) = \delta \vee 0 = \delta;$$
$$\text{If} \quad \mathbf{x} = (1,1) \quad \text{then } \Delta_{I_{\mathsf{E}_{T_P,S_P,N_S},T_P,N_S}}(\mathbf{x},\delta) = \delta \vee ((1-\delta+\delta^2)^2 - 1) = \delta;$$
$$\text{If} \quad \mathbf{x} = (0,1) \quad \text{then } \Delta_{I_{\mathsf{E}_{T_P,S_P,N_S},T_P,N_S}}(\mathbf{x},\delta) = \delta \vee 0 = \delta;$$
$$\text{If} \quad \mathbf{x} = (1,0) \quad \text{then } \Delta_{I_{\mathsf{E}_{T_P,S_P,N_S},T_P,N_S}}(\mathbf{x},\delta) = \delta \vee 0 = \delta^2(1 + (1-\delta)^2)^2.$$

The main contribution of this paper is concerned with the study of robustness on representable fuzzy X(N)or operators mainly used in fuzzy reasoning based on the product t-norm, the probabilistic sum and standard negation. Taking the class of Xor-implications E_{T_P,S_P,N_S}, the paper states the sensitivity of the $I_{\mathsf{E}_{T_P,S_P,N_S},T_P,N_S}$ fuzzy connective at the endpoints of U. The work of estimating its sensitivity to small changes is related to reducing sensitivity in the corresponding fuzzy connectives.

Our current investigation clearly aims the extension of the robustness studies to other main classes of (co)implications: S-(co)implications, R-(co)implications [4] and QL-(co)implications.

Keywords: *pointwise sensitivity, robustness, fuzzy Xor, fuzzy Xor-implication*

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