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Interval Newton's Method using Constrained Interval Arithmetic

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Abstract. This article presents the Newton's method for interval-valued functions, for this, is introduced a derivative concept to use the Taylor's theorem for interval-valued functions. The arithmetic structure associated with these results is the constrained interval arithmetic.

Keywords. Constrained Interval Arithmetic, Interval Taylor's Theorem, Interval Newton's Method

1 Introduction

As it is well known, the interval analysis has become an important tool to tackle problems whose mathematical models lead to some uncertainty in the parameters, due to the nature of the problem. Even though it has been much studied in the last decades, we can see that there is still a wide field to be studied, in this sense we can mention, for example, the interval functions (functions whose domain and image have interval elements) which has been little studied due to the difficulty of working with these, if we think about iterative processes of interval functions, we will see that we have a restricted amount of tools and techniques to address these, this in comparison with the interval-valued functions [4, 5, 8]. In this article, it is proposed to approach this problem using the constrained interval arithmetic (CIA, for short) proposed by Lodwick (see [6]), the same one that has been used in works like, [2, 3, 7], and apply this techniques to find the zeros of polynomial equations using Newton's method in the interval context.

This paper is organized as follows. In a preliminary section, we will look at some general aspects of the CIA, and will be given higher-order derivative and derivative concepts for interval-valued functions. In Section 3, we will proof Taylor's theorem in their interval version and from this in Section 4 we present Newton's method for solving problems involving the search for zeros of polynomial functions.

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2 Preliminaries

In this section we present some notations to be used throughout the article, and we hope that the reader will be familiar with the CIA.

We denote by,

$$\mathbb{I} = \{[\underline{a}, \overline{a}] \ : \ \underline{a} \leq \overline{a}, \underline{a}, \overline{a} \in \mathbb{R}\}$$

to set of all compact convex subsets of $\mathbb{R}$, this set will be called *interval space*. W. Lodwick in [6] define a linear representation of an interval $A = [\underline{a}, \overline{a}]$ as

$$A = \{\underline{a} + \lambda(\overline{a} - \underline{a}) \ : \ \lambda \in [0, 1])\},$$

where $\underline{a} + \lambda(\overline{a} - \underline{a})$ is called the constrained parametric representation associated to the interval A.

To simplify the notation we will write $\lambda, \lambda_1, \lambda_2, ...$ to denote the parameters associate to each interval. So, the constrained parametric representation of an interval A will be (see [1])

$$A = [\underline{a}, \overline{a}] = \{a(\lambda) = w_a\lambda + \underline{a} \ : \ \lambda \in [0, 1]\} = \{a(\lambda) = (\overline{a} - \underline{a})\lambda + \underline{a} \ : \ \lambda \in [0, 1]\}. \quad (1)$$

The algebraic operations for CIA are defined as follows. We consider two intervals $A = \{a(\lambda_1) \ : \ \lambda_1 \in [0, 1]\}$ and $B = \{b(\lambda_2) \ : \ \lambda_2 \in [0, 1]\}$, where $a(\lambda_1)$ and $b(\lambda_2)$ are the constrained parametric representation associated to the intervals A and B, respectively. Then

$$\begin{aligned} A \circ B &= C \\ &= [\underline{c}, \bar{c}] \\ &= \{a(\lambda_1) \circ b(\lambda_2) \ : \ \lambda_1, \lambda_2 \in [0, 1]\} \\ &= \{c \ : \ c = a(\lambda_1) \circ b(\lambda_2), \ \lambda_1, \lambda_2 \in [0, 1]\} \\ \text{where } \underline{c} = &\ \min\{c\}, \ \ \bar{c} = \max\{c\}, \ \ 0 \leq \lambda_1 \leq 1, 0 \leq \lambda_2 \leq 1 \\ \text{and } \circ &\ \in \{+, -, \times, \div\}. \end{aligned} \quad (2)$$

It is clear from (2) that constrained interval arithmetic is a constrained global optimization problem.

From CIA [6] we know that, for dependent operations, we consider the same constrained parametric representation for the same intervals involved in the algebraic opperations, i.e. $A \circ A = \{a(\lambda) \circ a(\lambda) \ : \ \lambda \in [0, 1]\}$, where $\circ \ \in \{+, -, \times, \div\}$.

Considering the notation used in G. Maqui et.al. [7], we define the next.

Definition 2.1. *[1] Let $f : \mathbb{R} \times \mathbb{R}^l \to \mathbb{R}$ be a function and let $c = (c_1, ..., c_l) \in \mathbb{R}^l$ be parameters involved with f. For each n-uple of intervals $\mathcal{C}^l$, we define a constrained parametric representation of $F_{\mathcal{C}^l}(x)$ by*

$$F_{\mathcal{C}^l}(x) = \left\{f_{c(\lambda)}(x) \ : \ f_{c(\lambda)} : \mathbb{R} \to \mathbb{R}, \ c(\lambda) \in \mathcal{C}^l\right\}. \quad (3)$$

Proposition 2.1. *[7] Let $f : \mathbb{R}\times\mathbb{R}^l \to \mathbb{R}$ be a continuous function in the second argument $c \in \mathbb{R}^l$. Then the interval-valued functions $F_{\mathcal{C}^l} : \mathbb{R} \to \mathbb{I}$ given by expression (3) is well defined and*

$$F_{\mathcal{C}^l}(x) = \left[\min_{\lambda\in[0,1]^l} f_{c(\lambda)}(x),\ \max_{\lambda\in[0,1]^l} f_{c(\lambda)}(x)\right], \tag{4}$$

for all $x \in \mathbb{R}$.

Note that if f is continuous in the second argument then the interval-valued function $F_{\mathcal{C}^l}$ is well defined and the interval $F_{\mathcal{C}^l}(x)$ is well defined (characterized) via its constrained parametric representation (3).

Next we will give a concept of derivative for an interval-valued function. This concept is based on the differentiability of each element of the constrained parametric representation.

Definition 2.2. *[7] Let $X \subset \mathbb{R}$ be an open set and let $F_{\mathcal{C}^l} : X \to \mathbb{I}$ be an interval-valued function. Suppose that $f_{c(\lambda)}$ is differentiable at x_0 for each $\lambda \in [0,1]^l$. Then we define the derivative of $F_{\mathcal{C}^l}$ at x_0, denoted by $F'_{\mathcal{C}^l}(x_0)$, by the constrained parametric representation*

$$F'_{\mathcal{C}^l}(x_0) = \left\{ f'_{c(\lambda)}(x_0)\ :\ c(\lambda) \in \mathcal{C}^l,\ \lambda \in [0,1]^l \right\}.$$

We say that $F_{\mathcal{C}^l}$ is differentiable at $x_0 \in X$ iff $F'_{\mathcal{C}^l}(x_0) \in \mathbb{I}$.

Proposition 2.2. *[7] Let $X \subset \mathbb{R}$ be an open set and let $F_{\mathcal{C}^l} : X \to \mathbb{I}$ be an interval-valued function. Suppose that $f_{c(\lambda)}$ is differentiable at x_0 for each $\lambda \in [0,1]^l$ and $f'_{c(\lambda)}(x_0)$ is continuous at λ. Then $F_{\mathcal{C}^l}$ is differentiable and*

$$F'_{\mathcal{C}^l}(x_0) = \left[\min_{\lambda\in[0,1]^l} f'_{c(\lambda)}(x_0)\ ,\ \max_{\lambda\in[0,1]^l} f'_{c(\lambda)}(x_0)\right]. \tag{5}$$

Analogously, if $f^{(n)}_{c(\lambda)}(x_0)$ is continuous in λ. For $F_{\mathcal{C}^l} : X \to \mathbb{I}$, the n-th derivative at x_0, denoted by $F^{(n)}_{\mathcal{C}^l}(x_0)$ is defined by their constrained parametric representation,

$$F^{(n)}_{\mathcal{C}^l}(x_0) = \left\{ f^{(n)}_{c(\lambda)}(x_0)\ :\ c(\lambda) \in \mathcal{C}^l,\ \lambda \in [0,1]^l \right\},$$

$$F^{(n)}_{\mathcal{C}^l}(x_0) = \left[\min_{\lambda\in[0,1]^l} f^{(n)}_{c(\lambda)}(x_0)\ ,\ \max_{\lambda\in[0,1]^l} f^{(n)}_{c(\lambda)}(x_0)\right].$$

We say that, the interval valued function $F_{\mathcal{C}^l}$ is a $C^n[a,b]$ function and we denote this by $F_{\mathcal{C}^l} \in C^n[a,b]$ if $F_{\mathcal{C}^l}$ is n times differentiable, and the function $F^{(n)}_{\mathcal{C}^l} : [a,b] \to \mathbb{I}$ is continuous.

For the criterion of stop of the method of Newton it is necessary to define an appropriate metric, in this sense, we will use the Pompeiu-Hausdorff metric H on $\mathbb{I}$, which is defined as:

Given the intervals $A = [\underline{a}, \overline{a}]$ and $B = [\underline{b}, \overline{b}]$,

$$H(A,B) = \max\{|\underline{a} - \underline{b}|, |\overline{a} - \overline{b}|\}. \tag{6}$$

3 Taylor's Formula for Interval-Valued Functions

In this section we introduce the Taylor's theorem for interval-valued functions $F_{\mathcal{C}^l}: X \to \mathbb{I}$ using the constrained interval arithmetic, for this, we consider their respective constrained parametric representation $f_{c(\lambda)}$, with $c(\lambda) \in \mathcal{C}^l$.

If we consider, $F_{\mathcal{C}^l} \in C^n[a,b]$, that $F_{\mathcal{C}^l}^{(n+1)}$ exists on $int([a,b])$ with their constrained parametric representation continuous in λ, and $x_0 \in [a,b]$. Then, we obtain the constrained parametric representation of $F_{\mathcal{C}^l}$, i.e. $f_{c(\lambda)},\ c(\lambda) \in \mathcal{C}^l$, by definition of a $C^n[a,b]$ function, we obtain all the conditions of the Taylor's Theorem for the real case. So, for every $x \in [a,b]$, there is a number $\xi(x)$ between x_0 and x with

$$f_{c(\lambda)}(x) = P_{(c(\lambda),n)}(x) + R_{(c(\lambda),n)}(x)$$

where, the nth Taylor's polynomial for $f_{c(\lambda)}$ about x_0 is:

$$\begin{aligned} P_{(c(\lambda),n)}(x) &= f_{c(\lambda)}(x_0) + f'_{c(\lambda)}(x_0)(x-x_0) + ... + \frac{f^{(n)}_{c(\lambda)}(x_0)}{n!}(x-x_0)^n \\ &= \sum_{k=0}^{n} \frac{f^{(k)}_{c(\lambda)}(x_0)}{k!}(x-x_0)^k \end{aligned}$$

and, their Lagrange remainder term associated to $P_n(x)$ is:

$$R_{(c(\lambda),n)}(x) = \frac{f^{(n+1)}_{c(\lambda)}(\xi(x))}{(n+1)!}(x-x_0)^{(n+1)}.$$

As the expressions $P_{(c(\lambda),n)}(x)$ and $R_{(c(\lambda),n)}(x)$ are continuous in λ then the minimum and maximum exist, then we obtain

$$\mathcal{P}_{(\mathcal{C}^l,n)}(x) = \left[\min_{\lambda\in[0,1]^l} P_{(c(\lambda),n)}(x),\ \max_{\lambda\in[0,1]^l} P_{(c(\lambda),n)}(x)\right] \tag{7}$$

called the nth Taylor interval polynomial associated for $F_{\mathcal{C}^l}$ about x_0, and

$$\mathcal{R}_{(\mathcal{C}^l,n)}(x) = \left[\min_{\lambda\in[0,1]^l} R_{(c(\lambda),n)}(x),\ \max_{\lambda\in[0,1]^l} R_{(c(\lambda),n)}(x)\right] \tag{8}$$

is called the Lagrange's interval remainder function associated to $\mathcal{P}_{(\mathcal{C}^l,n)}(x)$.

With the expressions (7) and (8),

$$F_{\mathcal{C}^l}(x) = \mathcal{P}_{(\mathcal{C}^l,n)}(x) + \mathcal{R}_{(\mathcal{C}^l,n)}(x).$$

Thus, we proved the following theorem.

Theorem 3.1 (Taylor's Theorem)**.** *Suppose $F_{\mathcal{C}^l} \in C^n[a,b]$, that $F_{\mathcal{C}^l}^{(n+1)}$ exists on $int([a,b])$ with their constrained parametric representation continuous in λ, and $x_0 \in [a,b]$. For every $x \in [a,b]$, there is a number $\xi(x)$ between x_0 and x with*

$$F_{\mathcal{C}^l}(x) = \mathcal{P}_{(\mathcal{C}^l,n)}(x) + \mathcal{R}_{(\mathcal{C}^l,n)}(x), \tag{9}$$

where $\mathcal{P}_{(\mathcal{C}^l,n)}$ and $\mathcal{R}_{(\mathcal{C}^l,n)}$ are given in the equations (7) and (8).

The equation (9) represents the Taylor's polynomial for $F_{\mathcal{C}^l}$ about x_0.

4 Newton's method for solution of interval equations

In this section we study the Newton's method based on Taylor's polynomials, for this. Suppose that $F_{\mathcal{C}^l} \in C^2[a,b]$. Let $x_0 \in [a,b]$ be an approximation to x such that $0 \notin F'_{\mathcal{C}^l}(x_0)$ and $|x - x_0|$ is small. Consider the first Taylor's polynomial for $F_{\mathcal{C}^l}(x)$ expanded about x_0 and evaluated at x.

$$F_{\mathcal{C}^l}(x) = \mathcal{P}_{(\mathcal{C}^l,1)}(x) + \mathcal{R}_{(\mathcal{C}^l,1)}(x),$$

and their constrained parametric representation is

$$f_{c(\lambda)}(x) = f_{c(\lambda)}(x_0) + f'_{c(\lambda)}(x_0)(x - x_0) + \frac{f''_{c(\lambda)}(\xi(x))}{2}(x - x_0)^2,$$

here $\xi(x)$ is between x and x_0, and $\lambda \in [0,1]^l$.

Since, $f_{c(\lambda)}(x) = 0$ for some $\lambda \in [0,1]$, the last equation gives

$$0 = f_{c(\lambda)}(x_0) + f'_{c(\lambda)}(x_0)(x - x_0) + \frac{f''_{c(\lambda)}(\xi(x))}{2}(x - x_0)^2.$$

Is well know that, the Newton's method is derived by assuming that since $|x - x_0|$ is small, then the term involving $(x - x_0)^2$ is much smaller, so:

$$0 \approx f_{c(\lambda)}(x_0) + f'_{c(\lambda)}(x_0)(x - x_0).$$

Solving for x gives

$$x \approx x_0 - \frac{f_{c(\lambda)}(x_0)}{f'_{c(\lambda)}(x_0)} = x_1.$$

This sets the stage for Newton's method, which stars with an initial approximation x_0 and generates the sequence $\{x_n\}_{n=0}^{\infty}$, by

$$x_{n+1} = x_n - \frac{f_{c(\lambda)}(x_n)}{f'_{c(\lambda)}(x_n)} \ \text{ for } n \leq 0, \ \lambda \in [0,1]^l.$$

Now, if we use the last iterative method, we obtain an expression that depends of λ, i.e. after the first iteration x_1 is an interval. This means that, we need an method for interval functions $\mathcal{F}_{\mathcal{C}^l} : \mathbb{I} \to \mathbb{I}$. The natural extension of the Newton's method considering the isotonicity property, is

$$X_{n+1} = X_n - \frac{\mathcal{F}_{\mathcal{C}^l}(X_n)}{\mathcal{F}'_{\mathcal{C}^1}(X_n)},$$

where $\mathcal{F}'_{\mathcal{C}^1}$ is an interval function.

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Example 4.1. *Suppose we wish to find some* $\overline{x} \in \mathbb{R}$, *such that* $0 \in F_{\mathcal{C}^1}(\overline{x})$, *with*

$$F_{\mathcal{C}^1}(x) = [1,3]x^2 - 2x.$$

Using the constrained parametric representation of $F_{\mathcal{C}^1}$, *we obtain*

$$f_{c(\lambda)}(x) = (1+2\lambda)x^2 - 2x,$$

their derivative is

$$f'_{c(\lambda)}(x) = 2(1+2\lambda)x - 2.$$

Then, for these functions the corresponding natural extension functions are $\mathcal{F}_{\mathcal{C}^l}$ *and* $\mathcal{F}'_{\mathcal{C}^1}$. *Considering* $x_0 = [\frac{1}{2}, \frac{11}{5}]$, *we obtain the next iterative results*

$n = 1:$ *Here* $m_\lambda(X_0) = (\frac{1}{2} + \frac{17}{10}\lambda_1) - \frac{(1+2\lambda_2)(\frac{1}{2}+\frac{17}{10}\lambda_1)^2 - 2(\frac{1}{2}+\frac{17}{10}\lambda_1)}{2(1+2\lambda_2)(\frac{1}{2}+\frac{17}{10}\lambda_1)-2}$, *then*

$$\begin{aligned} X_1 &= [\min m_\lambda(X_0), \max m_\lambda(X_0)], \\ X_1 &= [\tfrac{2}{3}, 2.1441]. \end{aligned}$$

$n = 2:$ *Here* $m_\lambda(X_1) = m_\lambda(X_0) - \frac{(1+2\lambda_2)(m_\lambda(X_0))^2 - 2(m_\lambda(X_0))}{2(1+2\lambda_2)(m_\lambda(X_0))-2}$, *then*

$$\begin{aligned} X_2 &= [\min m_\lambda(X_1), \max m_\lambda(X_1)], \\ X_2 &= [\tfrac{2}{3}, 2.00908]. \end{aligned}$$

If it is considered an error less than $0,15$ *in the iteration process, then it means that* $H(X_n, X_{n-1}) < 0,15$, *and for this example,* $H(X_2, X_1) = \max\{0; 0.13502\} < 0,15$, *and finally the all zeros of* $F_{\mathcal{C}^1}$ *are in* $[\frac{2}{3}, 2.00908]$.

5 Conclusions

In this article, it was established the Newton's method in the interval context making use of the constrained interval arithmetic, for this, it was necessary to establish the Taylor's Theorem in their interval version, as well as, to establish a stop criterion based on the Pompeiu-Hausdorff metric for the Interval Newton's method. An example was presented to validate the results.

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