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Influence of an Uncertain Unbalance in the Dynamics of Vertical Axis Washing Machines

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Abstract. A washing machine has an interesting and complex dynamical behavior, which can be well described by a set of nonlinear differential equations. When analyzing the dynamics of a washing machine, the steady state motion (periodic solution) is an important response to consider and can be evaluated as a solution of a periodic boundary-value problem. The unbalance generated by the unevenly distribution of clothes during centrifugation is highly random and, therefore, a stochastic model is necessary to take this characteristic into account. The novelty of this paper consists in the analysis of a washing machine dynamics considering the uncertainty in the unbalance. Therefore, a stochastic model is proposed for the dynamics of a washing machine. The steady state solutions are calculated using the shooting method combined with a sequential continuation to evaluate it across all the spin speeds of the machine. The probability distributions of the washing machine vibration at those different spin speeds are approximated using Monte Carlo simulations. The impact of the random unbalance in the vibration amplitude of the washing machine is also investigated.

Keywords. Nonlinear dynamics, Washing machine, Shooting method, Monte Carlo simulation

1 Introduction

Washing machines can be divided in two main categories: vertical or horizontal axis washing machines. An illustrative drawing of a vertical washing machine is presented in the Fig. 1, where its main components are highlighted. When considering the dynamics of washing machines, the analysis is usually restricted to a particular assemble of components called Washing Group (WG). A WG is composed by a hydraulic balancer, a drum, a tank and a drivetrain, and it is connected through a hang-suspension system to the cabinet of the washing machine.

This paper deals with the stochastic nonlinear dynamics [3] of a WG during centrifuge stage. When analyzing the dynamics of a WG, the random nature of the unbalance mass of clothes adds some significant difficulty. At each new washing cycle, the pieces of clothes move randomly during the washing phase and therefore become unevenly distributed around the drum during centrifuge. This uneven distribution generates a random unbalance mass, and therefore, must be incorporated in the model as random variables. Surprisingly, as far as the authors know, all the publications about the dynamics of vertical axis washing machines have considered the unbalance mass as a deterministic quantity, which is unrealistic.

In this paper, the main contribution consists in analyze the dynamics of a WG using a stochastic model [4], so that the random nature of the unbalance is taken into account. With this analysis, it is possible to investigate the impact of the uncertain unbalance in the dynamics of the WG, which is crucial to improve the reliability of new components during product development. The dynamics

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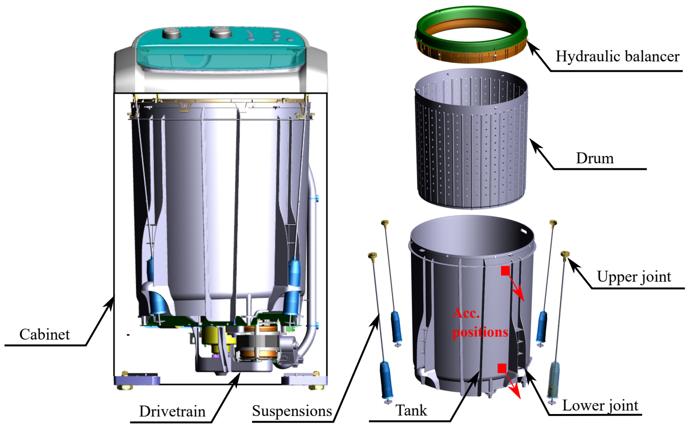

Figure 1: Illustration of a hang-suspension vertical axis washing machine. Fonte: [7].

of this stochastic model is analyzed in here through the computation of probability distributions of the WG's vibration (peak-to-peak amplitude) at different spin speeds. From those distributions of vibration levels, which considers the uncertainty in the unbalance parameters, it becomes possible to, for example, have a more precise procedure to evaluate the fatigue damage of the components. Also, it allows a proper set of gaps in the product to avoid possible impacts between the WG and the washing machine cabinet during centrifuge.

2 Equation of Motion

The equation of motion applied here to describe the WG dynamics was first presented by [2], and it was derived using a Lagrangian approach. In this model, it is assumed that all the components of the WG are rigid bodies, that the upper joints of the suspension system can not translate with respect to an inertial frame, and the suspension rods can not spin. Also, the inertial forces of the suspension's rods are neglected because of their small masses. Since this model describes the WG dynamics during centrifuge, it is assumed that all water from the washing phase have been drained out, leaving in the drum only wet clothes. A additional simplification of constant inertial parameter for the clothes are used.

To evaluate the equation of motion, two reference frames were used: $X_rY_rZ_r$, which is an inertial frame fixed to the ground, and $X_bY_bZ_b$, which is a local frame embedded in the tank. The X_b and Y_b axes are located in the plane that crosses all the lower spherical joints, and the Z_b axis is equal to the axis of rotation of the drum with respect to the tank. Both frames are schematically presented in Fig. 2.

2.1 Deterministic Model

Following the model proposed by [2], the equation of motion of the WG can be written as

$$(\mathbf{M}+\Delta\mathbf{M})\,\ddot{\mathbf{q}} = \frac{1}{2}\left[\frac{\partial \mathbf{M}}{\partial \mathbf{q}}\dot{\mathbf{q}}\right]^T \dot{\mathbf{q}} - \dot{\mathbf{M}}\dot{\mathbf{q}} + \mathbf{F}\left(\dot{\theta},\ddot{\theta}\right) + \mathbf{Q} + \mathbf{L} - \frac{\partial V_{WG}}{\partial \mathbf{q}}, \qquad (1)$$

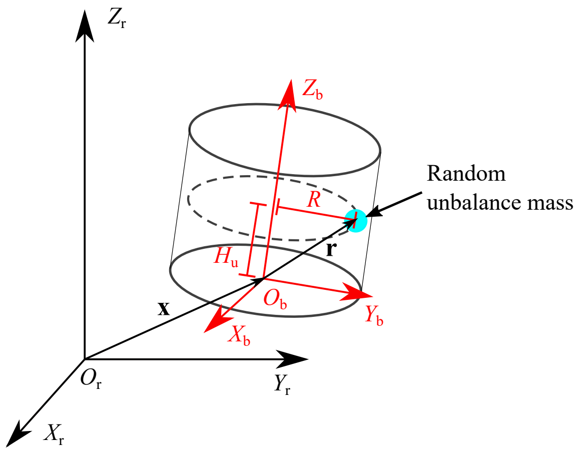

Figure 2: Schematic representation of the global and local frames. Characterization of the random unbalance through two random variables H_u and M_u. Fonte:[7].

where $\mathbf{q} = \begin{bmatrix} x & y & z & \alpha & \beta & \gamma \end{bmatrix}^T \in \mathbb{R}^6$ corresponds to the vector of generalized coordinates and it defines the position and orientation of the local frame with respect to the inertial frame. The matrices $\Delta\mathbf{M}$ and $\mathbf{M} \in \mathbb{R}^{6\times6}$ correspond to the mass matrix of the hydraulic balancer and the mass matrix of the rest of the WG's components, respectively. The vector $\mathbf{F}\left(\dot{\theta}, \ddot{\theta}\right) \in \mathbb{R}^6$ collects all the terms related to the spin speed $\dot{\theta}$ and spin acceleration $\ddot{\theta}$ of the drum. Vectors $\mathbf{Q}$ and $\mathbf{L} \in \mathbb{R}^6$ represent the generalized forces from the suspension system and from the hydraulic balancer, respectively. At last, V_{WG} represents the gravitational potential energy of the system. Any dot superscript represents a time derivative. Interested readers should resort for [1] and [2] for the complete derivation of this equation of motion.

2.2 Stochastic Model

In order to incorporate the random unbalance mass into the model, two continuous, uniform and independent random variables are first defined as:

$$\mathcal{U} \sim U[0.5,\ 1.5]\ \text{(kg)} \tag{2}$$

$$\mathcal{H} \sim U[0,\ 0.4]\ \text{(m)}, \tag{3}$$

where $\mathcal{U}$ is the random variable that defines the uncertain mass of the unbalance and $\mathcal{H}$ is the random variable that defines to height of the unbalance mass with respect to X_bY_b plane. The intervals for both uniform distributions were defined from experience and experimental observations of a global major manufacturer.

Details of how to derive the stochastic model of the WG including the effects of the random unbalance can be found in [7]. It is given by the following equation of motion:

$$\underbrace{(\mathbf{M} + \Delta\mathbf{M} + \mathbf{M}_u)}_{\mathbf{M}_T(\mathbf{q},\theta,\mathcal{U},\mathcal{H})} \ddot{\mathbf{q}} = \mathbf{f}_T\left(\mathbf{q}, \dot{\mathbf{q}}, \theta, \dot{\theta}, \ddot{\theta}, \mathcal{U}, \mathcal{H}\right), \tag{4}$$

where $\mathbf{q}$ is a random response for the washing machines dynamics since it depends directly on the random variable of the unbalance, $\mathcal{U}$ and $\mathcal{H}$. It consists in a nonautonomous mechanical system since θ is a known function of time.

3 Steady State Response

The steady state response of the WG vibration is now calculated. Therefore, the drum spin speed is considered constant and equals to Ω, so that it is possible to set $\ddot{\theta} = 0$, $\dot{\theta} = \Omega$ and $\theta = \Omega t$ in the equation of motion. The steady state response corresponds to the closed orbit found as the solution of the following periodic boundary-value problem:

$$\begin{cases} \dot{\mathbf{y}}(t) = \mathbf{g}\left(t, \mathbf{y}(t), \Omega, \mathcal{U}, \mathcal{H}\right), & \text{for } 0 \leq t \leq T \\ \mathbf{y}(0) = \mathbf{y}(T) \end{cases}, \tag{5}$$

where

$$\mathbf{g}\left(t, \mathbf{y}(t), \Omega, \mathcal{U}, \mathcal{H}\right) = \begin{bmatrix} \dot{\mathbf{q}}(t) \\ \mathbf{M}_T^{-1}\left(t, \mathbf{q}, \mathcal{U}, \mathcal{H}\right) \mathbf{f}_T\left(t, \mathbf{q}, \dot{\mathbf{q}}, \Omega, \mathcal{U}, \mathcal{H}\right) \end{bmatrix} \tag{6}$$

is the vector field, $\mathbf{y}(t) = \begin{bmatrix} \mathbf{q}^T(t) & \dot{\mathbf{q}}^T(t) \end{bmatrix}^T$ is the state of the system, and $T = \frac{2\pi}{\Omega}$ is the known period of the solution. The first line in Eq. (5) represents the same equation of motion defined in Eq. (4), rewritten in its state space form.

This periodic boundary-value problem can be solved numerically using the Shooting method [5]. This particular method searches for a specific initial state that, after the equation of motion is integrated along the known period, returns the system to the same initial state and therefore closes the orbit. To find this specific initial state, a residual vector $\mathbf{R}$ must be first defined as

$$\mathbf{R}(\mathbf{y}_0) = \mathbf{y}(T) - \mathbf{y}_0, \tag{7}$$

where $\mathbf{y}_0 = \mathbf{y}(0)$ is the specific initial state that represents the unknowns of the problem. Notice that the final state $\mathbf{y}(T)$ also depends on the specific initial state since it is obtained as the solution of an initial value problem with $\mathbf{y}_0$ as initial conditions. The solution of the periodic boundary value problem is found solving $\mathbf{R}(\mathbf{y}_0) = \mathbf{0}$, which can be done (within some error tolerance) using the Newton-Raphson solver. To integrate the equation of motion from the initial state to the final state, the 4th order Runge-Kutta method was used here. The Jacobian matrix $\frac{\partial \mathbf{R}(\mathbf{y}_0)}{\partial \mathbf{y}_0}$ required by the Newton-Raphson method was also computed numerically using a finite difference method.

It is important for the analysis discussed in this paper to evaluate the periodic solution of the WG vibration for the entire range of spin speeds of the machine. To this end, a sequential continuation was used [6]. The operational spin speed range was first defined from zero to the maximum spin speed, so that $\Omega \in [0,\ \Omega_{\max}]$. A discrete set of spin speed values, $\{\Omega_k\}_{k=0}^{N_s}$, was then defined dividing the operational spin speed range into equally spaced intervals, where $\Omega_k = k\frac{\Omega_{\max}}{N_s}$, and N_s is the number of intervals. The Shooting method was then used to solve the periodic boundary-value problem for each of those discrete values of spin speeds, sequentially, from low to high speeds. The first guessed solution for the periodic solution at a given spin speed Ω_k was set as the previous known solution at a spin speed Ω_{k-1}. The sequential continuation allows the periodic solutions to be define for the same values of discrete spin speed at every new simulation. This is an important requirement to evaluate the probability distribution of the WG vibration discussed in the next section and computed using Monte Carlo simulations. From experience, the sequential continuation should not face any problem while performing this continuation of periodic solutions since no bifurcation point is expected.

For each calculated periodic solution, the displacement of the tank was also calculated at two particular points, the top and bottom position, i.e., $\mathbf{s}_t$ and $\mathbf{s}_b$ respectively, as illustrated in Fig. 1. The top and bottom positions of the tank were chosen because they are common points used to attach accelerometers during vibration tests. For those two key displacements, $\mathbf{s}_t$ and $\mathbf{s}_b$, the analysis will be restricted to the peak-to-peak amplitude of the displacement in the X_r direction.

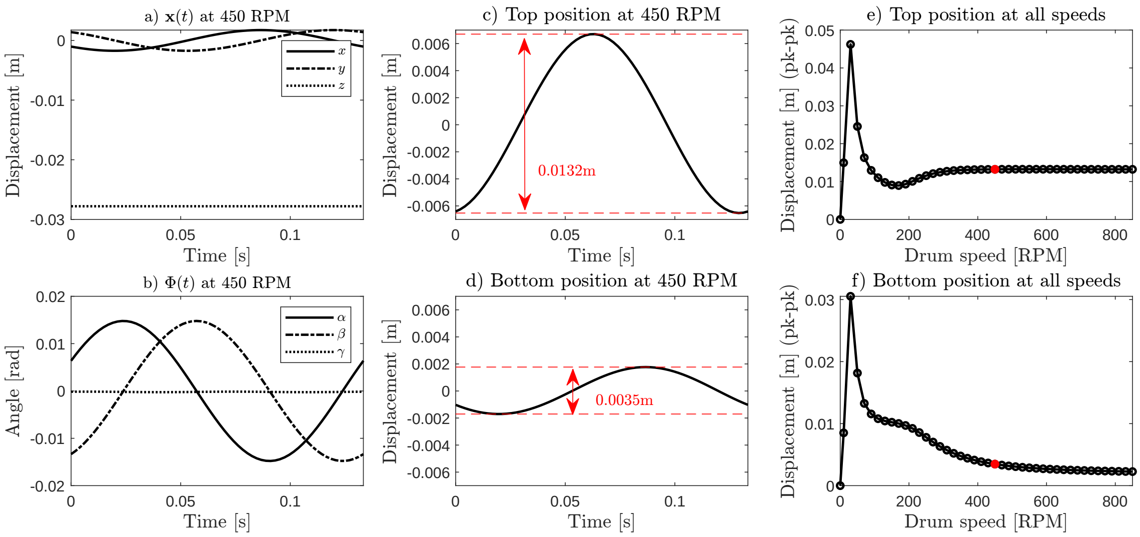

Figure 3: a) and b) represents the steady state response of the washer vibration. c) and d) are the radial displacements of the tank in the top and bottom position. e) and f) are the pk-pk top and bottom displacements of the tank for different spin speeds. Fonte:[7].

To illustrate this analysis procedure, a periodic solution of the WG computed using the Shooting method is presented in Fig. 3. For this particular simulation, a sample for of the unbalance mass and height was used ($\mathcal{U} = 0.576$kg and $\mathcal{H} = 0.335$m). In Fig. 3a, the translation of the local frame is presented, while its orientation (the Euler's angle) is presented in Fig. 3b. From this periodic solution, the displacements at the top and bottom of the tank in the X_r direction are presented in Fig. 3c and Fig. 3d, respectively. Repeating this procedure to all discrete predefined spin speeds, where the periodic solutions were calculated using the Shooting method and the sequential continuation, the peak-to-peak amplitude of the top and bottom displacements in the X_r direction can be plotted with respect to the spin speed, as showed by the in Fig. 3e and Fig. 3f. The red dots represent the peak-to-peak amplitude at 450 RPM. The steady state vibration curve can then be constructed connecting all the dots.

4 Monte Carlo Simulations

To analyze the probability distribution of the WG dynamics, Monte Carlo simulations were performed. A total of 10000 simulations were performed to estimate the probabilistic distribution of the WG vibration. Using all the computed steady state vibration curves, one histogram was constructed for each discrete spin speed (from 10 to 850 RPM with a 20 RPM increment). To this end, a total of 430000 periodic solutions of the stochastic model had to be calculated, which shows the computational cost of this analysis. With the implemented algorithm, a total of 1.15 s was required, on average, to compute each periodic solution on a personal computer. Figure 4a-f shows some of those histograms (for 150, 350 and 850 RPM). Figure 4g-h shows the concatenation of all histograms using color plots, where the dark red represents the highest probability, while the dark blue represents the lowest.

It is possible to notice that the histogram of the top displacements at 150 RPM (Fig. 4a) can be approximate by a uniform distribution. Meanwhile, all the other histograms with spin speed

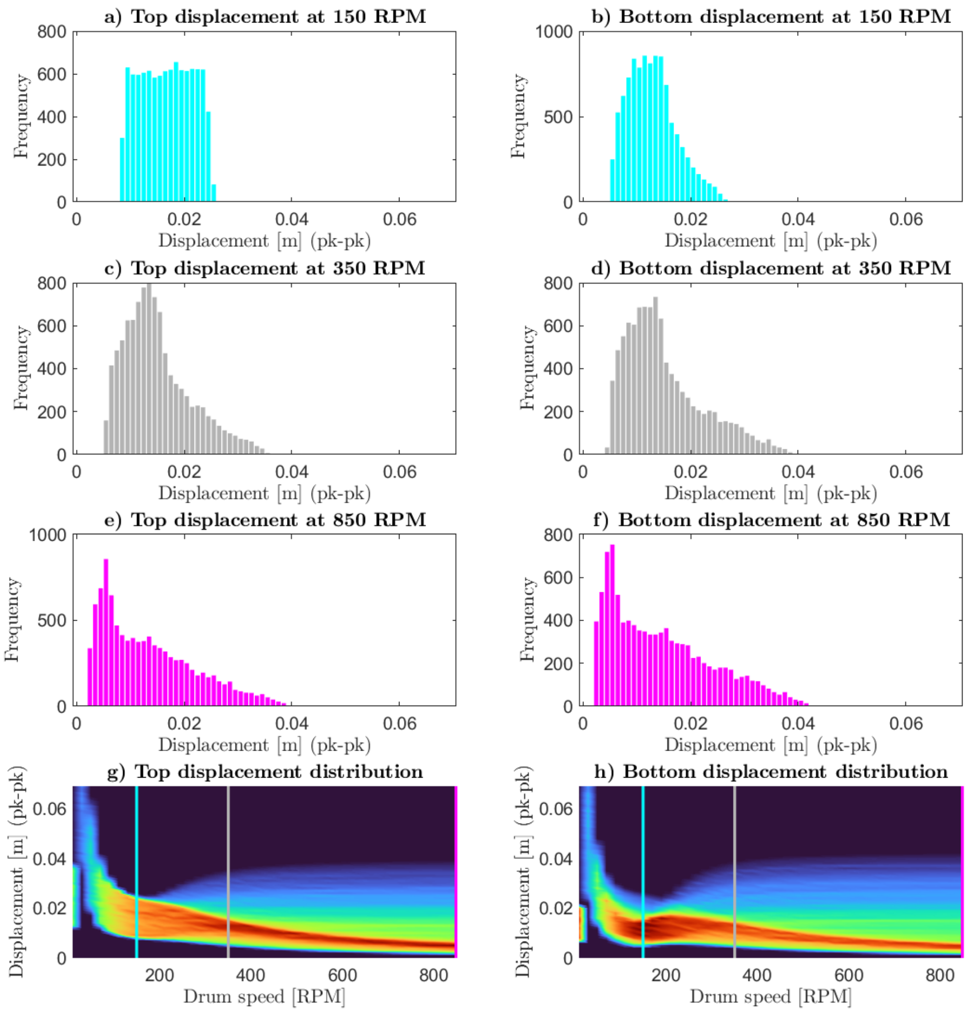

Figure 4: Probability distribution of the WG peak-to-peak vibration at different spin speeds. Fonte:[7].

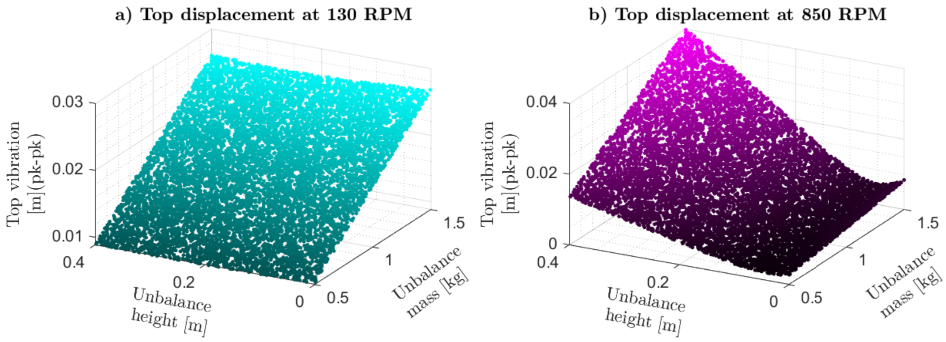

Figure 5: Scatter plot of the top displacement vibration amplitude (peak-to-peak) as function of the random variables samples. Fonte:[7].

above 200 RPM lose their symmetry with respect to the mean. This characteristic is enhanced as the spin speed increases.

To better understand this changing behavior in the distributions as function of the spin speed, two scatter plots are presented for two spin speeds, 130 and 850 RPM, as shown in Fig. 5. Both plots show the vibration amplitude (peak-to-peak) of the top displacement as function of the unbalance mass and height samples used in the Monte Carlo simulation.

5 Conclusions

From the results, it was observed a changing in the probability distribution of the WG vibration as function of the spin speed. At high speeds, the distributions became less symmetric. To better understand these changes in the distributions, a scatter plot of the vibration amplitude as function of the random variables were created. It was possible to conclude that at low spin speed, the vibration depends only on the unbalance mass and not on its height. As the spin speed becomes high, the unbalance height becomes also relevant in the vibration amplitude. At the end, a convergence analysis of some statistical moments of the vibration levels was conducted, and it was used to validated the amount of simulation runs used in the Monte Carlo simulation of this paper.

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