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Total Coloring Line Graphs of Generalized Petersen Graphs

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Abstract. A k-total coloring of G is an assignment of k colors to its elements (vertices and edges) such that adjacent or incident elements have distinct colors. The total chromatic number is the smallest integer k for which the graph G has a k-total coloring. If the total chromatic number is $\Delta(G)+1$, then G is called *Type 1*. The line graph of G, denoted by $L(G)$, is the graph whose vertex set is the edge set of G and two vertices of the line graph of G are adjacent if the corresponding edges are adjacent in G. In this paper, we prove that every line graph of the generalized Petersen graph is conformable; the graph $L(G(n, 1))$, for $n \geq 3$, is *Type 1*; and, if $L(G(n, k))$ is *Type 1*, then $L(G(n', k'))$ is *Type 1*, for all $n' \equiv 0 \mod n$ and $k' \equiv k \mod n$.

keywords. Total Coloring, Line Graph, Generalized Petersen Graph.

1 Introduction

Let $G = (V, E)$ be a simple connected graph. A *k-vertex coloring* of G is an assignment of k colors to the vertices of G so that adjacent vertices have different colors. A *k-edge coloring* of G is an assignment of k colors to the edges of G so that adjacent edges have different colors. The *chromatic index* of G, denoted by $\chi'(G)$, is the smallest k for which G has a k-edge coloring. Vizing's theorem states that the chromatic index $\chi'(G)$ is at least $\Delta(G)$ and at most $\Delta(G)+1$, where $\Delta(G)$ is the maximum degree of the graph G [10]. Graphs with $\chi'(G) = \Delta(G)$ are called *Class 1*, and graphs with $\chi'(G) = \Delta(G) + 1$ are called *Class 2*.

A *k-total coloring* of G is an assignment of k colors to the vertices and edges of G so that adjacent or incident elements have different colors. The *total chromatic number* of G, denoted by $\chi''(G)$, is the smallest k for which G has a k-total coloring. Clearly, $\chi''(G) \geq \Delta(G) + 1$ and the Total Coloring Conjecture (TCC) states that the total chromatic number of any graph is at most $\Delta(G) + 2$ [1, 10]. Graphs with $\chi''(G) = \Delta(G) + 1$ are called *Type 1*, and graphs with $\chi''(G) = \Delta(G) + 2$ are called *Type 2*.

A vertex coloring $\varphi : V(G) \to 1, 2, \ldots, \Delta(G) + 1$ is called *conformable* if the number of color classes (including empty color classes) of parity different from that of $|V(G)|$ is at most $def(G)$. Note that if G is a regular graph, then φ is called conformable if each color class has the same parity as $|V(G)|$. A graph is said to be *conformable* if it has a conformable vertex coloring; otherwise, it is said to be *non-conformable*.

The *generalized Petersen graph* $G(n, k)$, where $n \geq 3$ and $k \in \{1, 2, \ldots, n-1\}$, is the graph with vertex set $V(G(n, k)) = \{u_i, v_i \mid i \in 0, 1, \ldots, n-1\}$ and edge set $E(G(n, k)) = \{u_iu_{i+1}, u_iv_i, v_iv_{i+k} \mid i \in 0, 1, \ldots, n-1\}$, with indices taken modulo n. Clearly, the graph $G(n, k)$ and the graph $G(n, n-k)$ are isomorphic. Therefore, we can consider that $k \leq \lfloor \frac{n}{2} \rfloor$. Additionally, the graph $G(5, 2)$ is the well-known Petersen graph. The *line graph* of G, denoted by $L(G)$ is the graph whose

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vertex set is the edge set of G, and two vertices of $L(G)$ are adjacent if the corresponding edges are adjacent in G.

An important connection between the total chromatic number and the conformability of graphs was established in Theorem 1.1 and Corollary 1.1.

Theorem 1.1 (Chetwynd and Hilton [2], 1988)**.** *If G is Type 1, then G is conformable.*

Corollary 1.1 (Chetwynd and Hilton [2], 1988)**.** *If G is non-conformable, then G is not Type 1.*

It is known that determining the chromatic index and the total chromatic number are NP-complete problems even for regular graphs [6, 7]. In 2018, Vignesh et al. [9] conjectured that all line graphs of complete graphs $L(K_n)$ are *Type 1*. In 2021, Mohan et al. [8] verified the TCC to the set of quasi-line graphs, which is a generalization of line graphs, and present some infinite families of *Type 1* graphs. In 2022, Jayaraman et al. [5] determined the total chromatic number for certain line graphs. Recently, Faria et al. [4] determined the conformability of line graphs $L(G)$, when G is *Class 1* (Theorem 1.2) and proposed Question 1.1.

Theorem 1.2 (Faria et. al [4])**.** *Let G be a k-regular graph. If G is Class 1, then $L(G)$ is conformable.*

Question 1.1 (Faria et al. [4])**.** *Is there a k-regular graph G, $k \geq 3$, such that the line graph $L(G)$ is non-conformable?*

In 1969, Watkins [11] introduced the generalized Petersen graphs, an important graph class related to the well known Four color Conjecture. Watkins [11] established that the Petersen graph $G(5,2)$ is the only *Class 2* graph of the class. Dantas et al. [3] proved that there is a finite number of *Type 2* generalized Petersen graphs. Until now, the only known *Type 2* graphs of this class are $G(5,1)$ and $G(9,3)$.

In this paper, we prove that the line graph of the generalized Petersen graph is conformable; that $L(G(n,1))$, for $n \geq 3$, is *Type 1*; and prove that if $L(G(n,k))$ is *Type 1*, then $L(G(n',k'))$ is *Type 1*, for all $n' \equiv 0 \mod n$ and $k' \equiv k \mod n$. In order to investigate Question 1.1, we extend the search for *Type 2* line graphs from regular graphs which are conformable. Furthermore, we propose Conjecture 1.1, for which our results are positive evidences.

Conjecture 1.1. *If G is a k-regular, $k \geq 3$, Class 1 graph, then $L(G)$ is Type 1.*

2 Main Result

In this section, we prove that $L(G(n,k))$ is conformable, $L(G(n,1))$ is *Type 1* and we show that if $L(G(n,k))$ is *Type 1*, then $L(G(n',k'))$ is *Type 1*, where $n' \equiv 0 \mod n$ and $k' \equiv k \mod n$.

From the definition of line graphs, we describe $L(G(n,k))$ as follows: for $i \in \{0,1,\ldots,n-1\}$, the set of vertices of $L(G(n,k))$ is $u'_i = u_i u_{i+1}$, $m_i = u_i v_i$, and $v'_i = v_i v_{i+k}$, called the vertices of the *outer cycle*, the *articulations*, and the *inner cycles*, respectively. Therefore, the set of edges of $L(G(n,k))$ is $\{u'_i u'_{i+1}\} \cup \{v'_i v'_{i+k}\} \cup \{u'_i m_i, u'_i m_{i+1}\} \cup \{m_i v'_i, m_i v'_{i-k}\}$. Figure 1 presents three examples of $L(G(n,k))$.

Theorem 2.1. *The graph $L(G(n,k))$ is conformable.*

Proof. Consider the following two cases:

1. If $G(n,k)$ is the Petersen graph $G(5,2)$, then we exhibit a 5-total coloring for $L(G(5,2))$ (Figure 1c). Therefore, $L(G(5,2))$ is *Type 1*, and from Theorem 1.1, $L(G(5,2))$ is conformable.

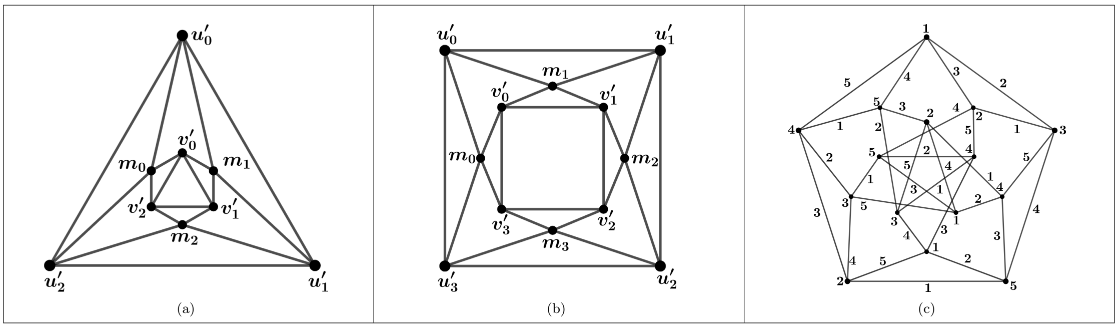

Figure 1: In 1a and 1b, the line graphs $L(G(3,1))$ and $L(G(5,1))$, respectively. In 1c, the line graph $L(G(5,2))$ with a 5-total coloring. Source: the authors.

2. If $G(n,k)$ is not the Petersen graph, then from Watkins [11], $G(n,k)$ is *Class 1*, and from Theorem 1.2, $L(G(n,k))$ is conformable.

□

Let $G_1 \simeq L(G(n_1,k))$ and $G_2 \simeq L(G(n_2,k))$ be two graphs with outer cycle, articulations, and inner cycles of G_1 denoted by u'_i, m_i, and v'_i for $i \in \{0,1,\ldots,n_1-1\}$, respectively, and of G_2 denoted by x'_i, p_i, and y'_i for $i \in \{0,1,\ldots,n_2\}$, respectively. The *gluing* operation of G_1 and G_2, denoted by $G_1 \leqq G_2$, results in the graph G, defined as follows:

$$V(G) = (V(G_1) \cup V(G_2)) \tag{1}$$

$$E(G) = [(E(G_1) \cup E(G_2)) \setminus R] \cup A \tag{2}$$

such that $R = R_1 \cup R_2$ and for each $i \in \{0,1,\ldots,k-1\}$:

$$R_1 = \{u'_0u'_{n_1-1}, m_0u'_{n_1-1}, m_iv'_{i+n_1-k}, v'_iv'_{i+n_1-k}\} \tag{3}$$

$$R_2 = \{x'_0x'_{n_2-1}, p_0x'_{n_2-1}, p_iy'_{i+n_2-k}, y'_iy'_{i+n_2-k}\} \tag{4}$$

and

$$\begin{aligned} A &= \{u'_0x'_{n_2-1}, u'_{n_1-1}x'_0, m_0x'_{n_2-1}, p_0u'_{n_1-1}\} \cup A' \\ A' &= \{m_iy'_{i+n_2-k}, p_iv'_{i+n_1-k}, v'_iy'_{i+n_2-k}, y'_iv'_{i+n_1-k}\} \end{aligned} \tag{5}$$

We call R the *set of edges removed for gluing* G_1 *with* G_2, and A the *set of edges added for gluing of* G_1 *and* G_2. Graph $G_1 \leqq G_2$ for $k=1$ is depicted in Figure 2.

Lemma 2.1, states that the gluing operation is closed for the class of line graphs of $G(n,k)$.

Lemma 2.1. *If $G_1 \simeq L(G(n_1,k))$ and $G_2 \simeq L(G(n_2,k))$, then $G_1 \leqq G_2 \simeq L(G(n_1+n_2,k))$.*

Proof. Let $G_1 := L(G(n_1,k))$ and $G_2 := L(G(n_2,k))$ with labeled vertices u'_i, m_i, and v'_i for G_1, and x'_i, p_i, and y'_i for G_2. It is easy to verify that the graph $G := G_1 \leqq G_2$ is 4-regular. To show that $G \simeq L(G(n_1+n_2,k))$, we will define a bijection r from G to $L(G(n_1+n_2,k))$ that preserves the adjacency between vertices and edges of G and $L(G(n_1+n_2,k))$.

Let $r : V(G) \to L(G(n_1+n_2,k))$ be defined as follows:

1. For the outer cycle:

$$\begin{aligned} r(u'_i) &= u'_i \quad \text{for } i \in \{0,1,\ldots,n_1-1\} \\ r(x'_i) &= u'_{n_1+i} \quad \text{for } i \in \{0,1,\ldots,n_2-1\} \end{aligned} \tag{6}$$

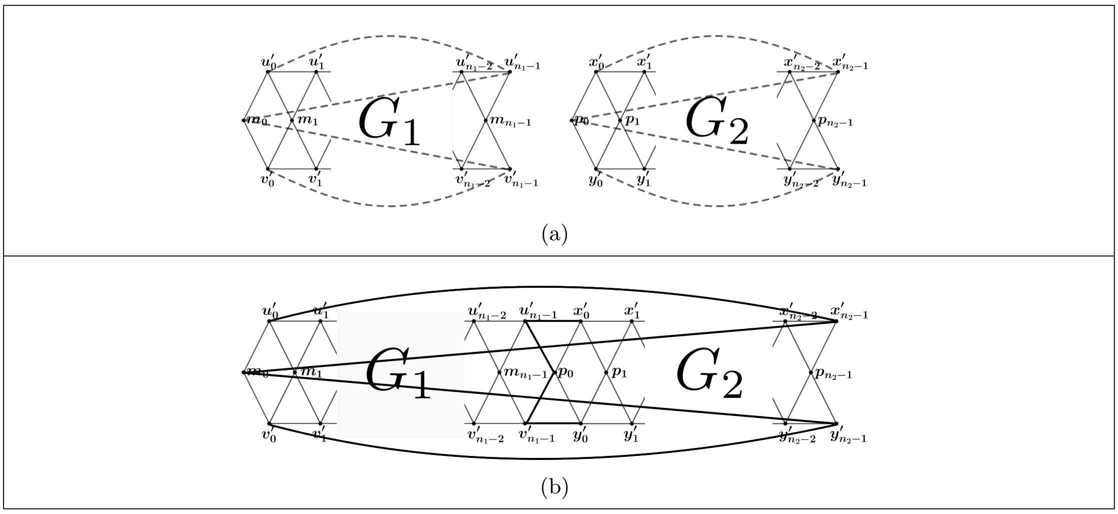

Figure 2: For $k = 1$, the gluing of G_1 with G_2 results in the graph G. In 2a, the dashed edges belong to the set $R_1 \cup R_2$. Note that the set R_1 corresponds to the dashed edges from the graph G_1, and the set R_2 corresponds to the dashed edges from the graph G_2. In 2b, the bold edges belong to the set A. Source: the authors.

2. For the articulations:

$$\begin{aligned} r(m_i) &= m_i \quad \text{for } i \in \{0, 1, \dots, n_1 - 1\} \\ r(p_i) &= m_{n_1+i} \quad \text{for } i \in \{0, 1, \dots, n_2 - 1\} \end{aligned} \tag{7}$$

3. For the inner cycles:

$$\begin{aligned} r(v_i') &= v_i' \quad \text{for } i \in \{0, 1, \dots, n_1 - 1\} \\ r(y_i') &= v_{n_1+i}' \quad \text{for } i \in \{0, 1, \dots, n_2 - 1\} \end{aligned} \tag{8}$$

If an edge ab in G does not belong to A, then it is easy to see that ab is an edge of G if and only if $r(a)r(b)$ is an edge of $L(G(n_1 + n_2, k))$. Suppose that $ab \in A$.

1. Suppose that $ab \in \{u_0' x_{n_2-1}', u_{n_1-1}' x_0', m_0 x_{n_2-1}', p_0 u_{n_1-1}'\}$. From (6) and (7):

   - $u_0' x_{n_2-1}'$ is an edge of G if and only if $r(u_0')r(x_{n_2-1}') = u_0' u_{n_1+n_2-1}'$ is an edge of $L(G(n_1 + n_2, k))$;
   - $u_{n_1-1}' x_0'$ is an edge of G if and only if $r(u_{n_1-1}')r(x_0') = u_{n_1-1}' u_{n_1}'$ is an edge of $L(G(n_1 + n_2, k))$.

   Hence, the vertices of G preserve the adjacency of elements in the outer cycle.

   - $m_0 x_{n_2-1}'$ is an edge of G if and only if $r(m_0)r(x_{n_2-1}') = m_0 u_{n_1+n_2-1}'$ is an edge of $L(G(n_1 + n_2, k))$;
   - $p_0 u_{n_1-1}'$ is an edge of G if and only if $r(p_0)r(u_{n_1-1}') = m_{n_1} u_{n_1-1}'$ is an edge of $L(G(n_1 + n_2, k))$.

   Hence, the vertices of G preserve the adjacency of elements in the outer cycle and articulations.

2. Suppose that $ab \in \{m_i y'_{i+n_2-k}, p_i v'_{i+n_1-k}, v'_i y'_{i+n_2-k}, y'_i v'_{i+n_1-k}\}$ such that $i \in \{0, 1, \ldots, k-1\}$. From (7) and (8):

   - $m_i y'_{i+n_2-k}$ is an edge of G if and only if $r(m_i)r(y'_{i+n_2-k}) = m_i v'_{i+n_1+n_2-k}$ is an edge of $L(G(n_1+n_2,k))$;
   - $p_i v'_{i+n_1-k}$ is an edge of G if and only if $r(p_i)r(v'_{i+n_1-k}) = m_{n_1+i} v'_{i+n_1-k}$ is an edge of $L(G(n_1+n_2,k))$.

   Hence, the vertices of G preserve the adjacency of elements in the articulations and inner cycles.

   - $v'_i y'_{i+n_2-k}$ is an edge of G if and only if $r(v'_i)r(y'_{i+n_2-k}) = v'_i v'_{i+n_1+n_2-k}$ is an edge of $L(G(n_1+n_2,k))$;
   - $y'_i v'_{i+n_1-k}$ is an edge of G if and only if $r(y'_i)r(v'_{i+n_1-k}) = v'_{n_1+i} v'_{n_1+i-k}$ is an edge of $L(G(n_1+n_2,k))$.

   Hence, the vertices of G preserve the adjacency of elements in inner cycles.

Therefore, $G \simeq L(G(n_1+n_2,k))$. □

Let ϕ_1 and ϕ_2 be two 5-total colorings of $L(G(n_1,k))$ and $L(G(n_2,k))$, respectively. We say that ϕ_1 is *compatible* with ϕ_2 if it is possible to define a 5-total coloring for $L(G(n_1+n_2,k))$ using ϕ_1 and ϕ_2.

Dantas et al. [3] proved that if $G(n,k)$ is *Type 1*, then $G(n',k')$ is *Type 1* for any $n' \equiv 0 \mod n$ and $k' \equiv k \mod n$. Similarly, we show that this property holds for the line graph of the generalized Petersen graph.

Theorem 2.2. *If $L(G(n,k))$ is Type 1, then $L(G(n',k'))$ is Type 1 for every $n' \equiv 0 \mod n$ and $k' \equiv k \mod n$.*

Proof. Let $n \geq 3$ and k be positive integers such that $L(G(n,k))$ is *Type 1* with a 5-total coloring ϕ. If $n' \equiv 0 \mod n$, then there exists a positive integer q_1 such that $n' = q_1 n$. First, we prove that $L(G(n',k))$ is *Type 1* and so we prove that $L(G(n',k'))$ is *Type 1*, where $k' \equiv k \mod n$. It is easy to see that $L(G(n',k))$ is obtained by gluing q_1 copies of $L(G(n,k))$. Therefore, to prove that $L(G(n',k))$ is *Type 1*, it is sufficient to show that ϕ is compatible with itself. From definition, gluing $L(G(n,k))$ to itself is obtained by removing the edges in (3) and (4) and adding the edges in (5), recursively, q_1 times. Moreover, from Lemma 2.1, $L(G(nq_1,k)) \simeq L(G(n',k))$ is obtained. Let us define a 5-total coloring ψ for $L(G(n',k))$.

1. If an element x of $L(G(n',k))$ does not belong to A, then we set $\psi(x) = \phi(x)$.
2. Suppose that x is an edge of $L(G(n',k))$ belonging to A. To distinguish the vertices between $L(G(n,k))$ and its copy, let u'_i, m_i, and v'_i be the labeled vertices of $L(G(n,k))$, and x'_i, p_i, and y'_i be the labeled vertices of its copy. We remark that $L(G(n,k))$ and its copy are assigned to the 5-total coloring ϕ, $\phi(u'_0) = \phi(x'_0)$, $\phi(u'_{n-1}) = \phi(x'_{n-1})$, and $\phi(m_0) = \phi(p_0)$. Thus, $\phi(u'_0) \neq \phi(x'_{n-1})$ and $\phi(m_0) \neq \phi(x'_{n-1})$. We assign, therefore:

   - $\psi(u'_0 x'_{n-1}) = \phi(u'_0 u'_{n_1-1})$;
   - $\psi(u'_{n-1} x'_0) = \phi(u'_0 u'_{n-1})$;
   - $\psi(m_0 x'_{n-1}) = \phi(m_0 u'_{n-1})$;
   - $\psi(p_0 u'_{n-1}) = \phi(m_0 u'_{n-1})$.

   For each $i \in \{0, 1, \ldots, k-1\}$, we have $\phi(m_i) = \phi(p_i)$ and $\phi(v'_{i+n-k}) = \phi(y'_{i+n-k})$. Thus, $\phi(m_i) \neq \phi(y'_{i+n-k})$ and $\phi(p_i) \neq \phi(v'_{i+n-k})$. Therefore, we assign:

- $\psi(m_i y'_{i+n-k}) = \phi(m_i v'_{i+n-k})$;
- $\psi(p_i v'_{i+n-k}) = \phi(p_i y'_{i+n-k})$;

Since $\phi(v'_i) = \phi(y'_i)$, $\phi(v'_i) \neq \phi(y'_{i+n-k})$ and $\phi(y'_i) \neq \phi(v'_{i+n-k})$. We assign, therefore, $\psi(v'_i y'_{i+n-k}) = \phi(v'_i v'_{i+n-k})$ and $\psi(y'_i v'_{i+n-k}) = \phi(y'_i y'_{i+n-k})$. Consequently, $L(G(n',k))$ is *Type 1*.

Let k' be a positive integer such that $k' \equiv k \mod n$, i.e., $k' = nq_2 + k$. Observe that $L(G(n',k'))$ can be obtained from $L(G(n',k))$ by replacing the edges $v'_i v'_{i+k}$ by $v'_i v'_{i+k+nq_2}$ and $m_i v'_{i-k}$ by $m_i v'_{i-k-nq_2}$. Additionally, in the gluing process, the vertex v'_{i+k+nq_2} belongs to the q_2-th copy of $L(G(n,k))$ and has the same assignment as the vertex v'_{i+k}, and the same holds for v'_{i-k-nq_2}.

Consequently, we can assign $\psi(v'_i v'_{i+k+nq_2}) = \phi(v'_i v'_{i+k})$ and $\psi(m_i v'_{i-k-nq_2}) = \phi(m_i v'_{i-k})$.

Therefore, $L(G(n',k'))$ is *Type 1*. □

Theorem 2.3. *The graph $L(G(n,1))$ is Type 1, for all $n \geq 3$.*

Proof. The proof is carried out as follows: we define 5-total colorings ϕ_1, ϕ_2, and ϕ_3 for the graphs $L(G(3,1))$, $L(G(4,1))$, and $L(G(5,1))$, respectively, as shown in Figures 3a, 3b, and 3c.

1. Suppose $n = 3q$. Observe that $L(G(3,1))$ is *Type 1*. From Theorem 2.2, $L(G(3q,1))$ is *Type 1*.
2. Suppose $n = 3q+1$. Observe that $L(G(3q+1,1))$ can be obtained by gluing $L(G(4,1))$ with $q-1$ copies of $L(G(3,1))$. Moreover, note that ϕ_2 is compatible with ϕ_1. Thus, $L(G(3q+1,1))$ is *Type 1*.
3. Suppose $n = 3q+2$. Observe that $L(G(3q+2,1))$ can be obtained by gluing $L(G(5,1))$ with $q-1$ copies of $L(G(3,1))$. Moreover, note that ϕ_3 is compatible with ϕ_1. Thus, $L(G(3q+2,1))$ is *Type 1*.

□

Corollary 2.1. *Let n and k be positive integers such that $n \geq 3$ and $k < \lfloor \frac{n}{2} \rfloor$. If $n' \equiv 0 \mod n$ and $k \equiv 1 \mod n$, then the graph $L(G(n',k))$ is Type 1.*

Proof. From Theorem 2.3 $L(G(n,1))$ is *Type 1* and from Theorem 2.2, there are two positive integers $q_1, q_2 \in \mathbb{N}$ such that $L(G(nq_1, nq_2+1))$ is *Type 1*. Thus, $L(G(n',k))$ is *Type 1* where $n' \equiv 0 \mod n$ and $k \equiv 1 \mod n$. □

3 Conclusion

In order to determine the total chromatic numbers of all $L(G(n,k))$ graphs for any pair of natural numbers (n,k) such that $k \leq \lfloor \frac{n}{2} \rfloor$, it is necessary to find compatible 5-total colorings among the graphs $L(G(2k+j,k))$ where $j \in \{1,2,\ldots,2k+1\}$, called *basic graphs*. Notice that for $k = 1$, the basic graphs are $L(G(3,1))$, $L(G(4,1))$, and $L(G(5,1))$, and Theorem 2.3 provides compatible 5-total colorings for them, allowing us to determine the total chromatic number of $L(G(n,1))$.

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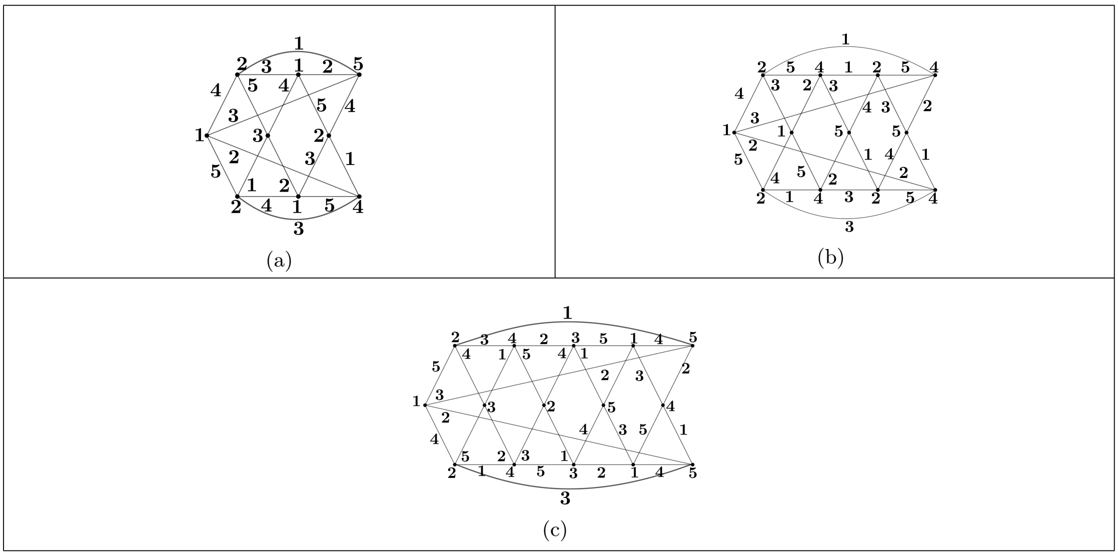

Figure 3: Three 5-total colorings ϕ_1, ϕ_2 and ϕ_3 to $L(G(3,1))$, $L(G(4,1))$ and $L(G(5,1))$, respectively. Source: the authors.

References

[1] M. Behzad. "Graphs and and their chromatic numbers". PhD thesis. Michigan State University, 1965.

[2] A. G. Chetwynd and A. J. W. Hilton. "Some refinements of the total chromatic number conjecture". In: *Congr. Numer.* (1988), pp. 195–216.

[3] S. Dantas, C. M. H. de Figueiredo, G. Mazzuoccolo, M. Preissmann, V. F. dos Santos, and D. Sasaki. "On the total coloring of generalized Petersen graphs". In: *Discrete Math.* 5 (2016), pp. 1471–1475.

[4] L. Faria, M. Nigro, and D. Sasaki. "On the conformability of regular line graphs". In: *RAIRO Oper. Res.* (2023).

[5] G. Jayaraman, D. Muthuramakrishnan, and S. Vishnu Kumar. "Total coloring of line graph and square graph for certain graphs". In: *Adv. Appl. Math. Sci.* 21.11 (2022).

[6] D. Leven and Z. Galil. "NP completeness of finding the chromatic index of regular graphs". In: *Journal of Algorithms* 4.1 (1983), pp. 35–44.

[7] C. J.H. McDiarmid and A. Sánchez-Arroyo. "Total colouring regular bipartite graphs is NP-hard". In: *Discrete Math.* 124.1 (1994), pp. 155–162.

[8] S. Mohan, J. Geetha, and K. Somasundaram. "Total coloring of quasi-line graphs and inflated graphs". In: *Discrete Math. Algorithms Appl.* 13.05 (2021).

[9] R. Vignesh, J. Geetha, and K. Somasundaram. "Total Coloring Conjecture for Certain Classes of Graphs". In: *Algorithms* 11.10 (2018).

[10] V. Vizing. "On an estimate of the chromatic class of a p-graph". In: *Metody Diskret. Analiz.* (1964), pp. 25–30.

[11] M. Watkins. "A theorem on Tait colorings with an application to the generalized Petersen graphs". In: *J. Comb. Theory* (1969), pp. 152–164.

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