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The Firefighter Game in Fullerene Graphs

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Abstract. In 1995, Hartnell introduced the Firefighter game with an input graph $G = (V, E)$. Initially, every vertex of V is labelled non-burned. The game consists of a fire starting at vertex $v \in V$ switching the label of v to burned. While there are non-burned vertices adjacent to burned vertices (the fire is still spreading), a non-burned vertex u is chosen to be defended, setting the label of u to defended. At each new step, the fire spreads from each burned vertex v to every non-burned adjacent vertices to v, which labels are set to burned. The game ends when the fire is no longer able to spread. We want to defend as many vertices as possible. Let $sn(G, v)$ be the maximum number of vertices that can be defended when a fire starts from a vertex $v \in V$. The *surviving rate* $\rho(G) = \frac{1}{|V|^2} \sum_{v \in V} sn(G, v)$ is the average percentage of vertices that can be defended when a fire starts from a vertex v of V. In this work, we apply a Firefighter game to fullerene graphs. We established $\rho(G)$, when G is the full icosahedral symmetry fullerene graph $G_{2,0}$.

Keywords. Cubic Graphs, Firefighter Game, Fullerene Graphs.

1 Introduction

A *graph* G is a ordered pair (V, E) consisting of a set V of *vertices* and a set E, disjoint from V, of *edges*, together with a incidence function φ_G that associates with each edge of G an unordered pair of (not necessarily distinct) vertices of G. If e is an edge and u and v are vertices such that $\varphi_G(e) = \{u, v\}$, vertices u and v are called the *ends* of e, and we say that these vertices are *adjacent* [2].

In 1995, Hartnell introduced the *Firefighter game* in graphs, which consists of a fire starting at one non-burned vertex of a graph and then a non-burned vertex is chosen to be defended. After the first defense, the *Step 1* is complete (the other steps follows the same). At each new step, the fire spreads from each burned vertex to all adjacent vertices that were not defended in the previous steps and, again, one vertex can be defended by firefighters, until the fire stops spreading [5]. The fire cannot burn or cross a defended vertex. We want to defend as many vertices as possible. In figures that we illustrate Firefighter strategies, red vertices represent the burned vertices and label $b_k, k \geq 1$, indicates the step at which the fire reached the respective vertex. Similarly, blue vertices represent the defended vertices and label $d_k, k \geq 1$, indicates the defense step. When the fire has just spread, black vertices represent indirectly defended vertices as the fire was contained. Figure 1 presents each step of an optimal strategy of Firefighter game on graph C_5.

Let $sn(G, v)$ denote the maximum number of vertices that can be defended when the fire starts at vertex v of graph G. The *surviving rate* $\rho(G) = \frac{1}{|V|^2} \sum_{v \in V} sn(G, v)$ is the average percentage of vertices that can be defended when a fire starts from a vertex v of V [3]. Note that in Figure 1, no matter where the fire starts, in an optimal strategy it is always possible to defend 3 vertices. Therefore, the survival rate of this graph is $\rho(C_5) = \frac{3+3+3+3+3}{5^2} = 60\%$.

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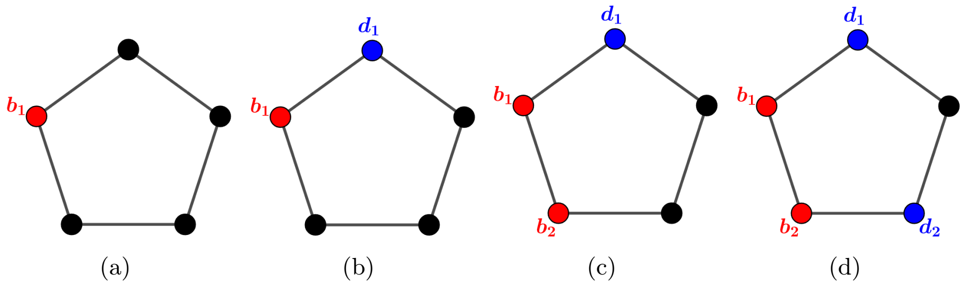


Figure 1: The 4 stages of Step 1 of Firefighter game on graph C_5, with an optimal strategy to defend 60% of it. In (a), the first burned vertex. In (b), the first defended vertex, when the first step is complete. In (c), the fire spreads to all adjacent vertices. In (d), the second defended vertex, when the second step is complete, and the fire can no longer spread. We have 3 of the 5 vertices defended. Source: From the authors.

One of the motivation of this game is the following. Suppose that some village of houses are represented by graphs. The houses are represented by the vertices and, if two houses are neighbors, the corresponding vertices have an edge connecting them. So, we fight fires in some village of houses trying to defend a maximum number of houses, saving the maximum number of houses that our strategy allows. This context seems to be appropriate when applied to large networks, where we can act and make decisions based on our own interests, as at each step of the game the player decides where to place a new firefighter on a non-burned vertex of the graph. By making this choice, the player is indirectly deciding which vertices can be set on fire in the next step, and consequently, which ones can be defended next. Our village of houses is represented by members of the class of *fullerene graphs*. In Figure 2, an example of a village of houses and your representation by a vertex labeled graph.

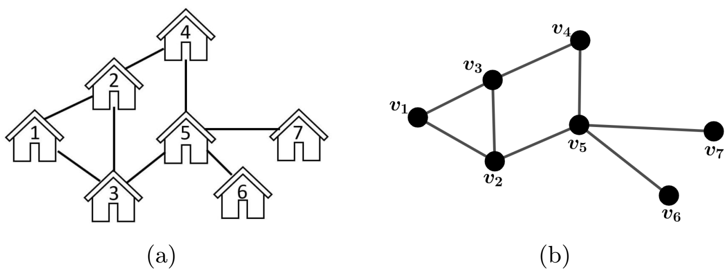


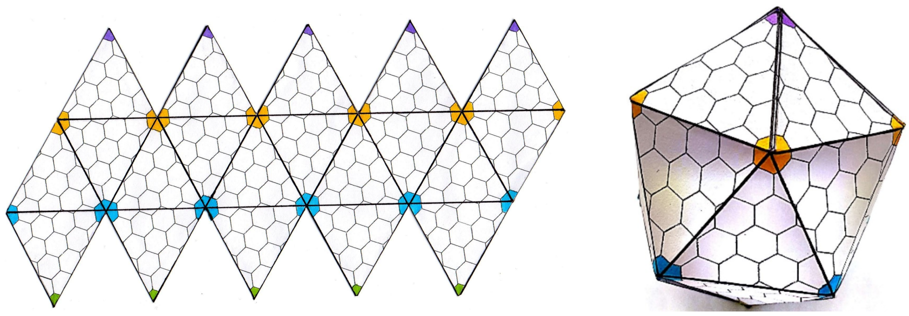
Figure 2: In (a), a village of houses. In, (b) the representation by a graph with labeled vertices. In this case the surviving rate is given by $\rho(G) = \frac{(5+4+4+4+4+5+5)}{7^2} = \frac{31}{49} \approx 63\%$. Source: From the authors.

In this work, we present computational results of the game applied to the smallest fullerene graphs, and introduce two strategies of the Firefighter game applied to the first two members of a full icosahedral symmetry fullerene family. We conclude this work by presenting a partial result on the third graph of this family.

2 Fullerene Graphs

The fullerene graphs are mathematical models of molecules allotropic to diamonds, composed only of carbon atoms containing 12 pentagonal faces and a non-specific number of hexagonal faces [1]. An important property of fullerene graphs is that they are *planar graphs*, that is, they have a representation on the plane so that the edges do not intersect, and therefore they resemble a village. These graphs are 3-connected, cubic, and planar.

The main result of this paper is to fight fires on the *full icosahedral symmetry fullerene graph* $G_{2,0}$, a graph of family $G_{i,0}, i \geq 2$, introduced by Andova and Škrekovski [1]. The first graph of this family $G_{1,0}$ is presented in Figure 6. In these graphs, the centers of the pentagonal faces form an icosahedron, based on the construction of its 20 triangular faces with the Goldberg vector in an hexagonal grid, a plane composed only of hexagonal faces. To illustrate the icosahedral symmetry fullerene graph $G_{1,4}$ and your construction, we refer to Figure 3.



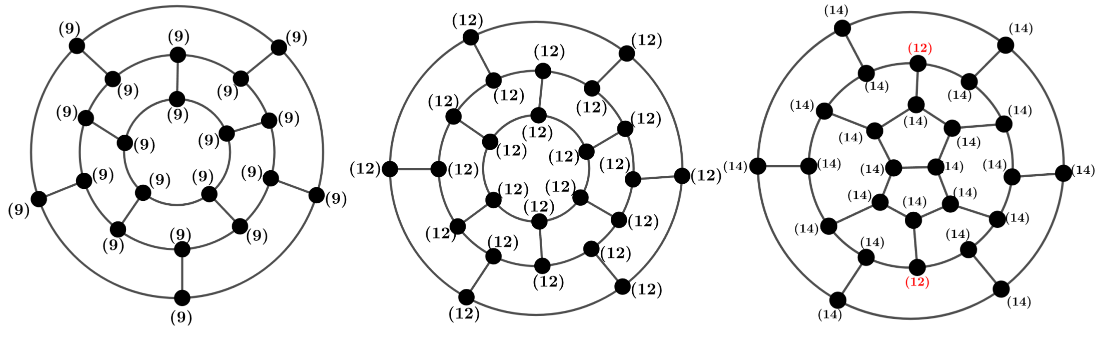
(a) (b)

Figure 3: In (a), the colored parts that will form the 12 pentagons of the solid with icosahedral symmetry generated by the vector $\vec{G} = (1, 4)$, and in (b) the solid with icosahedral symmetry generated by the vector $\vec{G} = (1, 4)$. On the surface of this solid, the icosahedral symmetry fullerene graph $G_{1,4}$ is defined [6]. Source: From the authors.

2.1 Computational Results on the Smallest Fullerene Graphs

With the help of the computer, we were able to simulate the game by analyzing all possible strategies, in order to find the maximum number of vertices defended in a given graph, with the fire starting at a random vertex. The program works as follows: we enter the number of vertices and edges in the graph, along with the adjacencies between the vertices. The program will scan the entire graph, analyzing what happens with each choice of initial vertex set on fire and with each defense successively, since as previously stated, at each step the player is indirectly deciding which vertices can be set on fire and defended in the next step. Thus, once the first vertex to be defended is chosen, the fire spreads and all possibilities of vertices that can be defended in the following steps will be analyzed. However, with each new choice, countless possibilities immediately arise, and these will also be calculated. After all possible scenarios, the program will make a new choice for the first defense, and consequently, all defense possibilities in the following steps. Then the program will carry out this entire analysis with a new vertex as first burned vertex, until all vertices are analyzed.

It is not difficult to see that in a graph with a large number of vertices the process becomes quite time-consuming. To illustrate, in Figure 4, we run the program at the three smallest fullerene graphs C_{20}, C_{24}, and C_{26}, and note that the first burned vertex influences the number of vertices that can be defended (represented in parentheses) at the end of the game. We remark that in the subsequent fullerene graphs from graph C_{26}, the vertices initially set on fire start to influence the maximum number of defended vertices.



(a) C_{20} (b) C_{24} (c) C_{26}

Figure 4: Fullerene graphs C_{20}, C_{24} and C_{26} with their respective numbers of vertices that can be defended from each vertex as the first burned vertex. Regarding graph C_{26}, with the majority of vertices being the first burned vertex, it is possible with an optimal strategy to defend 14 of the 26 vertices. But in two vertices as the first burned vertex, this number drops to 12. Source: From the authors.

3 Hexagonal Grid Strategy

Our strategy to defend fullerene graphs is based on the study of Firefighter on the infinite hexagonal grid, an infinite graph with exclusively hexagonal faces. Note that this grid is very similar to fullerene graphs, due to its planar with hexagonal faces, with the difference that it does not contain the 12 pentagonal faces in a fullerene graph. In this strategy, the infinite hexagonal grid is divided into 3 equal parts (3 blue axes). The defended vertices are on two of the three axes, on the right and on the left, and thus the fire is maintained on 1/3 of the grid, while 2/3 are defended. But this strategy cannot stop de spread of the fire, because the grid is infinite. The complete strategy can be seen in [4]. In Figure 5, we present an illustration of part of this strategy on the infinite hexagonal grid, saving 2/3 of it.

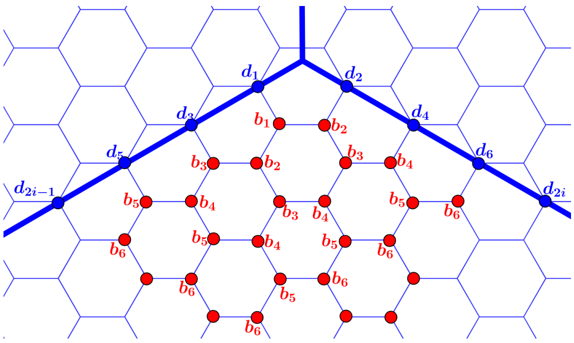


Figure 5: Our strategy in the hexagonal grid. The fire is maintained on 1/3 of the grid. Source: From the authors.

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Applying this same strategy to the smallest fullerene graph $G_{1,0}$, we do not obtain an optimal result, as we were able to verify what this result would be with the help of the computer.

The following algorithm gives the optimal result and an application of it can be seen in Figure 6: after sppliting the set of vertices of the graph into levels $C_i, i \geq 1$, the first defense must be a vertex from C_1. Defenses 2, 3 and 4 respect the condition $dist(c_2, d_1) = dist(c_3, d_2) = dist(c_3, d_1) = 2$, respectively. The last defense will be the vertex with distance equal to the diameter from the first burned vertex. Thus, the surviving rate of this graph is 45%, a possible lower bound for the surviving rate of all fullerene graphs, as this one is the smallest of them. In this context, we propose the following conjecture.

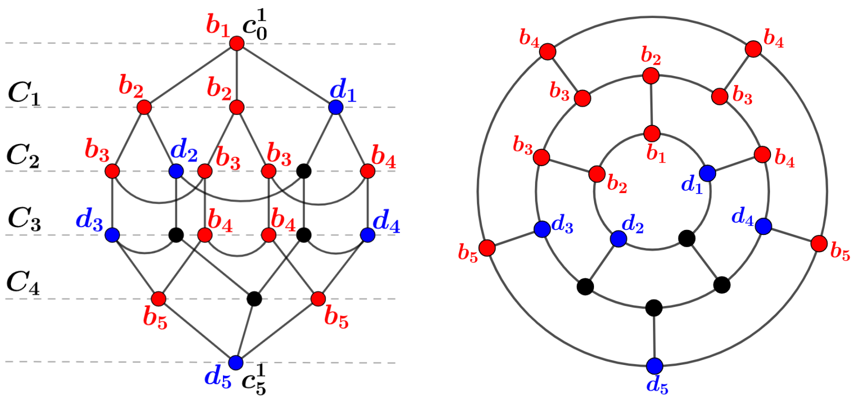


Figure 6: Application of the algorithm that gives the optimal result for graph $G_{1,0}$. Source: From the authors.

Conjecture 3.1. *Every fullerene graph has a surviving rate of at least* 45%*.*

3.1 Hexagonal Grid Strategy on Fullerene Graph $G_{2,0}$

The use of computational assistance on graph $G_{2,0}$ (Figure 7) had a very large computational cost, due to its high number of vertices. So let us analyze two cases: when the first burned vertex is a pentagonal vertex (a vertex that belongs to at least one pentagonal face) and when the first burned vertex is an exclusively hexagonal vertex (a vertex that belongs only to hexagonal faces). Each graph $G_{i,0}$ of the family contains $20i^2$ vertices, so in $G_{2,0}$ we have 80 vertices. In this graph, all exclusively hexagonal vertices have distance 1 from the nearest pentagonal vertex.

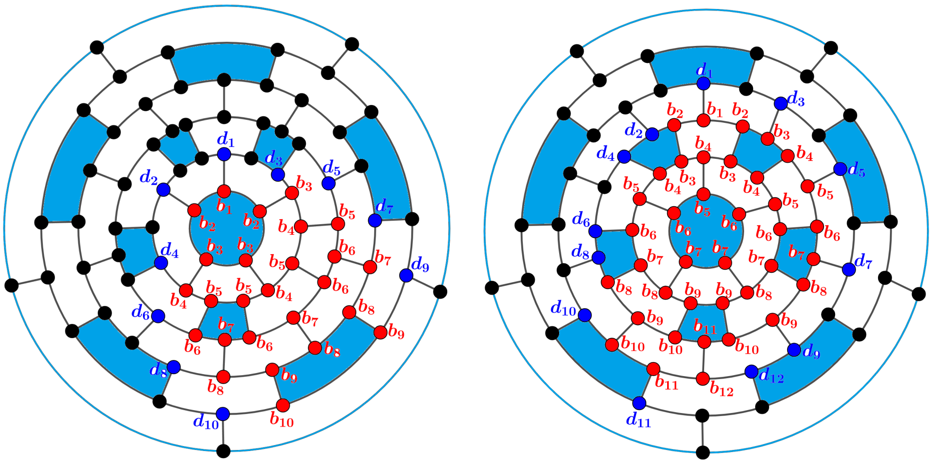
In Figure 7, we have the application of the infinite hexagonal grid strategy in the graph $G_{2,0}$, in the two cases we are considering: the fire starting at a vertex belonging to a pentagonal vertex and the case in that the fire starts at an exclusively hexagonal vertex. Note that when applying this strategy in the case where the fire starts at an hexagonal vertex, we obtain 43 defended vertices. In the case where the fire starts at a pentagonal vertex, we obtain 54 defended vertices. As the graph $G_{2,0}$ has 60 pentagonal vertices and 20 exclusively hexagonal vertices, we can then, using this strategy, calculate its survival rate, which is $\rho(G_{2,0}) \geq \frac{(60 \cdot 54)+(20 \cdot 42)}{80^2} \approx 64\%$.

When the fire starts at a pentagonal vertex, we analyze the number of burned vertices at each step on $G_{2,0}$ and on $G_{3,0}$ (Figure 8), and it is possible to see a pattern. Thus, in this case, we are able to predict the number of vertices defended in any graph $G_{i,0}$, thus having an upper bound for the number of vertices defended in the entire family. With this strategy, the number of burned vertices of $G_{i,0}$ in step i can be found in Equation 1, and the number of defended vertices of $G_{i,0}$

in step i can be found in Equation 2. Note that for i sufficiently large, the number of defended vertices is no more than 90%.

$$4\sum_{n=3}^{i+1} i + 2(i+5) = 2i^2 + 8i + 2 \tag{1}$$

$$\rho(G_{i,0}) \geq \frac{20i^2 - (2i^2 + 8i + 2)}{20i^2} = \frac{9}{10} - \frac{2}{5i} - \frac{1}{10i^2}, i \geq 2\cdot \tag{2}$$



(a) (b)

Figure 7: Strategy that defends 64% of fullerene graph $G_{2,0}$: In (a), the first burned vertex is a pentagonal vertex, returning a surviving rate of 67,5% (54 defended vertices). In (b), the first burned vertex is an hexagonal vertex, returning a surviving rate of 53% (42 defended vertices). Therefore, the surviving rate is $\rho(G_{2,0}) \geq \frac{(60\cdot 54)+(20\cdot 42)}{80^2} \approx 64\%$. Note that when first burned vertex is a pentagonal vertex, we defend a large number of vertices. The analysis of this observation will be investigated in the future. Source: From the authors.

4 Future Works

We did a brief analysis of Firefighter in the next graph of this family, $G_{3,0}$, when the fire starts at a pentagonal vertex. Observing the number of vertices defended, from this first burned vertex, we see that this number tends to grow as the family grows. In the case of $G_{3,0}$, which has 180 vertices, applying the same strategy as the hexagonal grid we managed to defend 136 vertices. Part of this ongoing work is to analyze the cases of fire starts in hexagonal vertices, of which we have two types: hexagonal vertices with distance 1 and distance 2 to a nearest pentagonal vertex.

As the construction of these graphs is done respecting the distance between the 12 pentagons according to the index i, an interesting fact is that, with the first burned vertex being a pentagonal vertex, we observed a pattern of vertices on fire in both of $G_{2,0}$ and $G_{3,0}$, and we established the number of burned vertices in any $G_{i,0}$, and thus also the number of vertices defended.

We will investigate whether there is also a regularization of the number of burned vertices, applying this strategy when the first burned vertex is an exclusively hexagonal vertex, thus establishing the number of defended vertices in the entire family regardless of where the fire starts.

In Figure 8, we have the strategy on $G_{3,0}$ with a first burned vertex being a pentagonal vertex. Note that the proportion of defended vertices $(75,5\%)$ increases in relation to $G_{2,0}$ $(67,5\%$, in Figure 7a), and we believe that in each subsequent graph of this family, a greater number of defended vertices is reached. And of course, in the future we will also analyze the strategy when the first burned vertex is a exclusively hexagonal vertex, trying to find a pattern for this case as well.

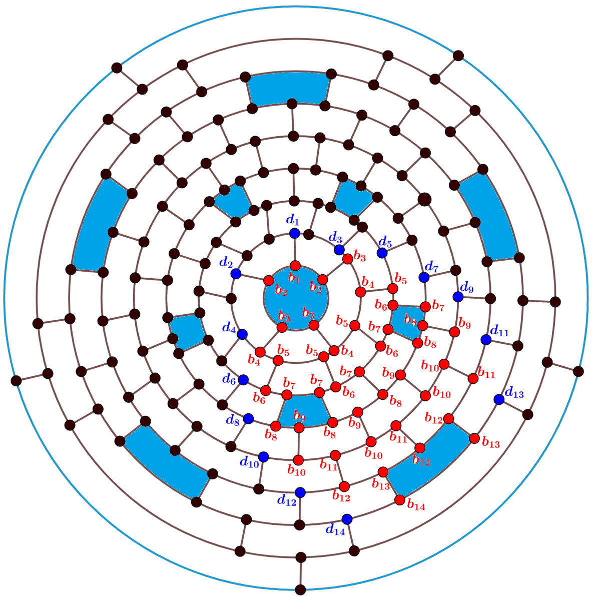


Figure 8: The same strategy applied on $G_{3,0}$, saving $75,5\%$ of the vertices, with a first burned vertex being a pentagonal vertex. Source: From the authors.

References

[1] V. Andova and R. Škrekovski. "Diameter of Fullerene Graphs with Full Icosahedral Symmetry". In: **MATCH Communications in Mathematical and Computer Chemistry** 70 (2013), pp. 205–220.

[2] J. Bondy and U. Murty. **Graph Theory**. New York: Springer Publishing Company Incorporated, 2008.

[3] L. Cai and W. Wang. "The Surviving Rate of a Graph for the Firefighter Problem". In: **Discrete Math** 23 (2009), pp. 1814–1826.

[4] T. Gavenčiak, J. Kratochvíl, and P. Prałat. "Firefighting on square, hexagonal, and triangular grids". In: **Discrete Mathematics** 337 (2014), pp. 142–155.

[5] B. Hartnell. "Firefighter! An application of domination." In: **The 25th Manitoba Conference on Combinatorial Mathematics and Computing**. 1995.

[6] D. Nicodemos. "Diâmetro de grafos fulerenes e transversalidade de ciclos ímpares de fuleróides-(3, 4, 5, 6)". PhD thesis. Universidade Federal do Rio de Janeiro, 2017.

