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Parametric Linear Stability Analysis of Tethered System Dynamics in Keplerian Orbits

Denilson P. S. Santos[1]
Universidade Estadual Paulista (UNESP), FESJ, São João da Boa Vista, SP
José L. M. Neto[2]
Universidade Federal da Paraíba (UFPB), Rio Tinto, PB
Jorge K. S. Formiga[3]
Universidade Estadual Paulista (UNESP), ICT, São José dos Campos, SP

Abstract. The objective of this study is to analyze the dynamics of a tether system comprised of two point masses interconnected by a cable (Space Tethers), orbiting a Newtonian center of attraction in a Keplerian orbit without external forces. Through reductions in the equations of motion, a Hamiltonian function is derived, and four stationary solutions are identified, two of which are stable. The study delves into the parametric linear stability concerning the eccentricity parameter (e) of the elliptical orbit and another parameter denoted as α, representing the angle between the tether's projection and the orbit plane. By employing the Deprit-Hori method alongside numerical computations, the study maps stability and instability regions in the parameter plane $\alpha \times e$.

keywords. Stability, tether system, dumbbell dynamics, Deprit-Hori method, orbital debris

1 Introduction

The concept of using a long, robust cable to connect objects in space, such as satellites, spacecraft, or space stations, to either the Earth's surface or to each other, is not a new idea. However, the engineering work required to implement this project was deemed impossible in the mid-twentieth century, when implementation attempts began. Currently, the challenges associated with constructing such structures still persist [1, 3, 4, 7, 8].

The first successful tether mission was the TSS-1R mission, launched by NASA in 1996. This mission deployed a 20-kilometer-long tether from the Space Shuttle, which was used to study the dynamics of tethered systems in space. Since then, there have been several other successful tether missions, including the European Space Agency's (ESA) Proba-2 mission in 2009 and the Japanese Aerospace Exploration Agency's (JAXA) Kounotori 6 mission in 2016.

Space tethers have a variety of potential applications, including space debris removal [5], satellite deployment and retrieval [2], fuel optimization in orbital maneuvers for spacecraft [9]. However, there are still many technical challenges that need to be overcome before space tethers can be widely used. These challenges include developing materials that are strong enough to withstand the harsh space environment and designing systems that can handle the forces and stresses involved in tethered operations.

[1]denilson.santos@unesp.br
[2]laudelino@dcx.ufpb.br
[3]jorge.formiga@unesp.br

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In Santos' analysis [10] concerning the dynamics of a tethered system, equilibrium conditions were scrutinized for a configuration comprising a dumbbell structure, wherein two point masses are interconnected by a cable of negligible mass and variable length. The motion within a planar elliptical orbit is delineated relative to the system's center of mass. The equations of motion were derived utilizing Lagrangian mechanics, encompassing the kinetic and potential energies of the system. Control laws governing the rotation angle around the center of mass were formulated, and stability conditions were investigated using Floquet theory. Additionally, Santos [10, 11] extends the analysis to encompass three-dimensional motion beyond the reference plane for the dumbbell system.

This study aims to investigate the stability of uniformly rotating configurations of a dumbbell interconnected by cables within a Newtonian central force field exhibiting Keplerian planar motion. The system comprises two infinitesimal point masses orbiting elliptically around the primary body, experiencing solely the gravitational force without any external influences. Utilizing the Hamiltonian dynamic system, equilibrium points will be analyzed, and the linear stability conditions of the tethered system will be determined.

Moreover, as an illustrative application of this theoretical framework, the concept of a capture network, also known as a space hub, will be explored. A capture network involves interconnecting multiple spacecraft or satellites using tether systems to form a cohesive structure in space. By analyzing the stability of such configurations, insights can be gained into the feasibility and practical implementation of space hubs for various purposes, such as satellite servicing, orbital assembly, and distributed sensing networks [6]. This application demonstrates the versatility and relevance of the theoretical analysis in addressing real-world challenges and advancing space exploration and technology.

2 Modeling the System

The Hamiltonian of a system is a mathematical function that characterizes the total energy of the system. In the context of space tethers, the Hamiltonian incorporates the kinetic energy of the masses tethered together, along with the potential energy associated with the tether itself, considering factors such as deformation or bending. This perspective complements prior investigations [11], which focused on analyzing the Lagrangian of tether systems from various aspects. Our approach involves examining the Hamiltonian of the dynamic system and assessing its linear stability concerning minor oscillations in orbital eccentricity. Additionally, the Hamiltonian accounts for any external forces acting on the system, including gravitational, aerodynamic, and electromagnetic forces. The analyzed system consists of two point masses (m_1 and m_2) connected to each other by a cable of negligible mass, forming a halter-type system, in a Keplerian orbit around the primary in reference frame with no external forces acting on the system.

To obtain the equations of motion of the system, the Lagrangian formulation is used, which relates the potential and kinetic energies and the generalized coordinates of the masses, this method avoids the need to know all the forces acting on the system and simplifies the mathematical analysis. Its general equation is given by Eq. 1.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j, \tag{1}$$

where q_j are the generalized coordinates; by hypothesis no external forces are acting on the system other than gravitational attraction, thus it is considered that $Q_j = 0$, since the total energy of the system is constant.

The Lagrange Equations of motion are ordinary differential equations, which describe the motions of mechanical systems under the action of forces, can be obtained by $L = T - V$. Where T

is the kinetic energy and V is the potential energy.

For the following analysis, the generalized coordinates are φ and ψ. The system is only under the gravity-gradient. These forces make a conservative system. Two equations can be obtained based on those coordinates, where $()' = \frac{d}{d\nu}$ utilizing the following transformation, as the motion is assumed to be in a Keplerian orbit around the primary $\frac{d}{dt} = \dot{\nu}\frac{d}{d\nu} = \omega_0(1 + e\cos(\nu))^2\frac{d}{d\nu}$.

$$\begin{aligned}\cos\psi\Big[4(e\cos\nu + 1)l'(\varphi' + 1)\cos\psi + l\Big(2\varphi''\cos\psi(1 + e\cos\nu)\\ - 4(e\cos\nu + 1)(\varphi' + 1)\psi'\sin\psi + \cos\psi(3\sin(2\varphi) - 4e\sin\nu(\varphi' + 1))\Big)\Big] = 0\end{aligned} \tag{2}$$

$$\begin{aligned}8(1 + e\cos\nu)l'\psi' + l\Big[2\varphi'\sin(2\psi)(2 + \varphi')(1 + e\cos\nu)\\ + \sin(2\psi)\Big(5 + 2e\cos\nu + 3\cos(2\varphi)\Big) + 4(1 + e\cos\nu)\psi'' - 8e\psi'\sin\nu\Big] = 0\end{aligned} \tag{3}$$

To find the equilibrium positions, the terms $\varphi' = 0, l' = 0$ and $\psi' = 0$ are replaced in Eqs. 2 and 3, to obtain

$$\varphi'' = \frac{4e\sin\nu - 3\sin(2\varphi)}{2(1 + e\cos\nu)} \tag{4}$$

$$\psi'' = -\frac{\sin(2\psi)(5 + 2e\cos\nu + 3\cos(2\varphi))}{4(1 + e\cos\nu)} \tag{5}$$

The conditions to the equilibrium positions are obtained for $\varphi'' = 0$ and $\psi'' = 0$, also being able to refer to them as static solutions, so the equilibrium happens when

$$\psi = \frac{k\pi}{2},\ k = 0, 1, 2, \ldots, \qquad \sin(2\varphi) = \frac{4}{3}e\sin\nu. \tag{6}$$

Because of the symmetry of the problem, we can restrict the values of k to $0, 1, 2$ and 3. For high eccentricities (e), close to 1, prohibitive regions appear that do not have equilibrium solutions, when $\psi = \frac{\pi}{2}$, the equilibrium is dependent of e and ν and they are continuous and periodic for $e \leq 0.75$. When the eccentricity (e) exceeds this value, solutions are not found for the true anomaly using this approach.

$$\begin{aligned}\Big\{\varphi(\nu) \to \boxed{\tfrac{1}{2}\left(-\sin^{-1}\left(\tfrac{4}{3}e\sin(\nu)\right) + 2\pi c_1 + \pi\right)\ \text{if } c_1 \in \mathbb{Z}}\Big\}\\ \Big\{\varphi(\nu) \to \boxed{\tfrac{1}{2}\left(\sin^{-1}\left(\tfrac{4}{3}e\sin(\nu)\right) + 2\pi c_1\right)\ \text{if } c_1 \in \mathbb{Z}}\Big\}\Big\}\end{aligned} \tag{7}$$

2.1 Hamiltonian Function

The focus will be Eq. 5 with $\cos(2\varphi) = \alpha,\ -1 \leq \alpha \leq 1$. Then, it turns into

$$\psi'' + \frac{5 + 2e\cos\nu + 3\alpha}{4(1 + e\cos\nu)}\sin(2\psi) = 0. \tag{8}$$

Since α depends on time ν, what we call equilibrium positions are, in fact, stationary solutions. Also, beside α evolves with time, as a dynamical variable, the study is done for fixed values of $\alpha \in [-1, 1]$, so we see α as a parameter.

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Let $p_\psi = \psi'$, then equation of motion in (8) is represented by an Hamiltonian system

$$\psi' = \frac{\partial H}{\partial p_\psi}, \qquad p'_\psi = -\frac{\partial H}{\partial \psi},$$

with Hamiltonian function given by

$$H(\psi, p_\psi) = \frac{1}{2}p_\psi^2 - \frac{5 + 2e\cos\nu + 3\alpha}{8(1 + e\cos\nu)}\cos(2\psi). \tag{9}$$

With the assumptions made so far, $\psi' = 0, \varphi' = 0$ and $l' = 0$, by studying the Hamiltonian function H around the stationary solutions $\psi = \frac{k\pi}{2}$, $(k = 0, 1, 2, 3)$ and $p_\psi = 0$, with $-1 \leq \alpha \leq 1$, this means that we investigate the linear stability conditions for the stationary solutions in the angle ψ associated with the parameters α and e, since $\alpha = \cos(2\varphi)$ and the angle φ must attend the conditions of Eq. 6.

In the case of stationary solutions $\psi_0 = \pi/2, (3\pi)/2$ and $p_{\psi 0} = 0$, the rod of the dumbbell is perpendicular to the plane of the orbit. Let,

$$\hat{x} = \psi + \psi_0, \qquad \hat{y} = p_\psi + p_{\psi 0},$$

then the linearized Hamiltonian function in (9) around these stationary solutions are the same, and given by

$$H(\hat{x}, \hat{y}) = \frac{1}{2}\hat{y}^2 - \frac{5 + 3\alpha + 2e\cos\nu}{4(1 + e\cos\nu)}\hat{x}^2. \tag{10}$$

The case of circular orbit, when $e = 0$, these stationary solutions are unstable for all values of $\alpha \in [-1, 1]$, because the characteristic polynomial of the 2×2 matrix JG_1 has real roots, where

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and G_1 is the Hessian matrix of the Hamiltonian function in (10) for $e = 0$.

For the equilibrium position $\psi_0 = 0, \pi$ and $p_{\psi 0} = 0$, the rod of the dumbbell is in the plane of the orbit. By linearizing the Hamiltonian function in (9) around these stationary solutions, yields the same Hamiltonian function

$$H(\hat{x}, \hat{y}) = \frac{1}{2}\hat{y}^2 + \frac{5 + 3\alpha + 2e\cos\nu}{4(1 + e\cos\nu)}\hat{x}^2. \tag{11}$$

For the case e equal to zero, these stationary solutions are linearly Lyapunov stable for all the values of $-1 \leq \alpha \leq 1$, as the characteristic polynomial of the matrix JG_2 has only imaginary roots, $\pm i\sqrt{(5 + 3\alpha)/2}$, where G_2 is the Hessian matrix of $H(\hat{x}, \hat{y})$ in (11) when $e = 0$.

3 Conclusion

Beginning with the challenge of managing a tether in a Keplerian orbit around a Newtonian gravitational center, upon simplification, we arrive at the equations of motion (2) and (3). These equations involve variables such as φ, denoting the rotation angle of the tether around its center of mass within the orbital plane; ψ, representing the angle of elevation relative to the orbital plane; the eccentricity parameter of the orbit, e; and the true anomaly, ν.

Stationary solutions occur when $\psi = (k\pi)/2$ $(k = 0, 1, 2, 3)$ and $\sin(2\varphi) = (4/3)e\sin\nu$. Introducing a parameter α dependent on the angle φ, and employing a Hamiltonian function pertinent to the problem, we investigate the stability conditions concerning ψ, with respect to α and e.

Notably, for $\psi = \pi/2$ and $\psi = (3\pi)/2$, the stationary solutions prove to be unstable. However, for $\psi = 0$ and $\psi = \pi$, the stationary solutions demonstrate stability. In the scenario of stable stationary solutions, we delve into the conditions of linear stability by delineating curves that demarcate regions of stability and instability in the parameter plane of $\alpha \times e$. Analytically, we compute these curves for sufficiently small values of e, while resorting to numerical methods for arbitrary values of e.

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4 References

[1] V. S. Aslanov, A. S. Ledkov, A. K. Misra, and A. D. Guerman. "Dynamics of Space Elevator After Tether Rupture". In: **Journal of Guidance Control and Dynamics** 36 (2013), pp. 986–992. ISSN: 15333884. DOI: 10.2514/1.59378.

[2] V. S. Aslanov, A. K. Misra, and V. V. Yudintsev. "Chaotic attitude motion of a low-thrust tug-debris tethered system in a Keplerian orbit". In: **Acta Astronautica** 139 (2017), pp. 419–427. ISSN: 0094-5765. DOI: https://doi.org/10.1016/j.actaastro.2017.07.008.

[3] V. V. Beletski and E. M. Levin. "Dynamics of space tether systems." In: **Published for the American Astronautical Society by Univelt San Diego** (1993), p. 449.

[4] M. P. Cartmell and D. J. McKenzie. "A review of space tether research". In: **Progress in Aerospace Sciences** 44 (1 Jan. 2008), pp. 1–21. ISSN: 0376-0421. DOI: https://doi.org/10.1016/j.paerosci.2007.08.002. URL: http://www.sciencedirect.com/science/article/pii/S0376042107000656.

[5] Z. Guang and Z. Jing-rui. "Space Tether Net System for Debris Capture and Removal". In: **4th International Conference on Intelligent Human-Machine Systems and Cybernetics** 1 (2012), pp. 257–261. DOI: 10.1109/IHMSC.2012.71.

[6] W. Huang, H. Zou, H. Liu, W. Yang, J. Gao, and Z. Liu. "Contact dynamic analysis of tether-net system for space debris capture using incremental potential formulation". In: **Advances in Space Research** 72.6 (2023), pp. 2039–2050. ISSN: 0273-1177. DOI: https://doi.org/10.1016/j.asr.2023.05.054.

[7] K. Kumar and K. D. Kumar. "Tethered dual spacecraft configuration: A solution to attitude control problems". In: **Aerospace Science and Technology** 4 (2000), pp. 495–505. ISSN: 12709638. DOI: 10.1016/S1270-9638(00)01064-6.

[8] A. K. Misra, Z. Amier, and V. J. Modi. "Attitude dynamics of three-body tethered systems." In: **Acta Astronautica** (1988), p. 1059.

[9] A. F. B. A. Prado. "Engineering notes using tethered gravity-assisted maneuvers for planetary capture". In: **Journal of Guidance, Control, and Dynamics** 38 (2015), pp. 1852–1855. ISSN: 15333884. DOI: 10.2514/1.G001009. URL: https://arc.aiaa.org/doi/10.2514/1.G001009.

[10] D. P. S. Santos and A. Ferreira. "Three-dimensional Two-Body Tether System — Equilibrium Solutions". In: **Journal of Physics: Conference Series** 641.1 (2015), p. 012009. DOI: 10.1088/1742-6596/641/1/012009. URL: https://dx.doi.org/10.1088/1742-6596/641/1/012009.

[11] D. P. S. Santos, J. L. M. Neto, V. T. Azevedo, and J. K. S. Formiga. "Linear stability analysis in tether system using its Hamiltonian function". In: **The European Physical Journal Special Topics** 232 (2023), pp. 3175–3183. ISSN: 1951-6355. DOI: 10.1140/epjs/s11734-023-01022-0.

