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Fractional Euler-Bernoulli Model: Dimensional Analysis and Applications

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Abstract. A generalized fractional solution, presenting dimensional analysis, is proposed for the Euler-Bernoulli beam model in the case of a simply supported beam subjected to a uniformly distributed static load. This solution is used to evaluate and identify the shear effects, covered by the Timoshenko-Ehrenfest solution, as well as the effects on the deflection of the stress concentration in the supports of the aforementioned beam. The shear effects are evaluated by adjusting the parameter α incorporated in the fractional Euler-Bernoulli solution with the Timoshenko-Ehrenfest solution. The evaluation of the effects of supports is done by adjusting the fractional Euler-Bernoulli solution with the solution obtained from a linear analysis carried out in the ANSYS software. The adjustment, in both cases, is made by taking the maximum deflection as a reference.

Keywords. Euler-Bernoulli, Timoshenko-Ehrenfest, ANSYS, solução fracionária

1 Introduction

In structural engineering, the classical theories Euler-Bernoulli (EB) and Timoshenko-Ehrenfest (TE) are of fundamental importance for the study of deflection in a linear elastic regime. These models, however, rely on specific restrictions. The EB beam theory models with considerable quality, beams with its cross-section a height h smaller than width b, that is, thin beams. Furthermore, the case of small deflections is also imposed, so that shear effects to ensure negligible. The TE beam theory, on the other hand, contemplates the effects of shear, thus being broader than the EB. This theory considers with relevant quality the deflection in tall beams, that is, beams with a h greater than b [9], also restricted to simple and linear systems.

Due to the simplicity of the aforementioned models, numerous instability effects, which arise in beams when subjected to different stresses, are not considered. The concentration of stress in the support regions, instability due to buckling, and the effect of large deflections are examples of effects not considered in such the models.

Aiming to consider the aforementioned instabilities, the fractional Euler Bernoulli (FEB) and fractional Timoshenko-Ehrenfest (FTE) models are here proposed. The development of such models for a simply supported beam with uniformly distributed load and the evaluation and detection of different instabilities, such as shear effects, stress concentration at supports, large deflections, buckling and plasticities are presented in [3]. The evaluation and detection of shear effects and

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instabilities related to stress concentration in supports and large deflections in said beams were published in [6] and [4]. In the FEB and FTE models presented in [3] the dimensions were disregarded, as the objective was to present a new tool for a better modeling. This work aims to propose an FEB model for a simply supported beam with uniformly distributed load, considering dimensional analysis, and apply it to evaluate and detect the effects of shear, covered by TE, and the effects of stress concentration on the supports.

It is worth remembering for the study of the evaluation of stresses in supports, the numerical solutions obtained using ANSYS [1] software are used as a reference basis. As the software provides data referring to the simulations without explaining each effect individually, and the objective is to use the fractional solution to evaluate each effect separately, of this work, the effects were initially filtered of shear of the solution obtained in the ANSYS simulation. In this case, the filtering process consisted of subtracting from the ANSYS solution the residues obtained from the difference between the TE solution and the EB solution, which represents the shear effects.

2 Analytical Development

For this study, a prismatic, homogeneous, isotropic, linear elastic, simply supported beam under a uniformly distributed load, with length L and rectangular cross section, as defined in [5]. In [3] a generalized fractional model FEB was proposed starting from the classical Euler-Bernoulli model. This model, which did not consider dimensional analysis, was used to evaluate several physical effects, including stress concentration in the supports. Here, the model (FEB) will be presented considering the dimensional analysis, as developed in the next section.

2.1 Fractional Euler-Bernoulli Solution with Dimension

According to [7], the classical EB model is given as follows

$$EI\frac{d^4v(x)}{dx^4} = -q(x), \tag{1}$$

where E is the Young's modulus and I is the rectangular moment of inertia of the cross section. You can rearrange the terms of Equation (1) and write it as follows.

$$\frac{d^4v(x)}{dx^4} = -\frac{q(x)}{EI}. \tag{2}$$

It can be seen that this model is well sized, as on the left side of Equation (2) the unit is $\frac{1}{m^3}$, corresponding to the unit obtained on the right side. From Equation (2) we obtain

$$v(x) = \frac{q}{EI}\left(\frac{-x^4}{24} + \frac{Lx^3}{12} - \frac{L^3x}{24}\right), \tag{3}$$

whose unit of deflection $v(x)$ is the meter. The fractional model proposed by [3], in which the dimension was not analyzed, is the following:

I) For the fractional parameter α in the interval $3 < \alpha \leq 4$

$$\frac{d^\alpha v(x)}{dx^\alpha} = -A, \tag{4}$$

with $A = \frac{q}{EI}$ and the boundary conditions $v(0) = v(L) = 0$, $EIv''(0) = M(0) = 0$ and $EIv''(L) = M(L) = 0$, where L is the length of the beam and M is the bending moment.

The left side of the Equation (4) has the unit $\frac{m}{m^{\alpha}} = m^{1-\alpha}$. The right side has the unit m^{-3}. Therefore, to equate the two members of the Equation (4), the left side must be multiplied by a constant whose unit must be $m^{\alpha-4}$. Denoting such constant by K and inserting it into the model, we obtain

$$K^{\alpha-4}\frac{d^{\alpha}v(x)}{dx^{\alpha}} = -A, \tag{5}$$

which is equivalent to

$$\frac{d^{\alpha}v(x)}{dx^{\alpha}} = -\frac{A}{K^{\alpha-4}}. \tag{6}$$

Taking $\overline{A} = \frac{A}{K^{\alpha-4}}$, we obtain the scaled model

$$\frac{d^{\alpha}v(x)}{dx^{\alpha}} = -\overline{A}. \tag{7}$$

It can be seen that K has the meter as its unit. Therefore, we can define it as a function of the length L of the beam, or as a function of x, a variable that describes the variation in deflection along the length L of the beam, when it is subjected to a load. We opted for K being a function of L, as the analytical solution of the model becomes much simpler. To solve Equation (7), the Laplace transform of the Caputo derivative was applied, as defined in [3]. By doing this, we obtain

$$\begin{aligned} L\left(\frac{d^{\alpha}v}{dx^{\alpha}}\right) &= -\overline{A}L(1) \\ s^{\alpha}V(s) - s^{\alpha-1}v(0) - s^{\alpha-2}v'(0) - s^{\alpha-3}v''(0) - s^{\alpha-4}v'''(0) &= -\overline{A}\ \frac{1}{s}. \end{aligned} \tag{8}$$

From the boundary conditions for the problem, we have $v(0) = 0$. Hence, taking $v'(0) = k_1$, $v''(0) = k_2$ and $v'''(0) = k_3$ with k_1, k_2 and k_3 constants, substituting in Equation (8) and rearranging the terms, we obtain

$$V(s) = -\overline{A}\ \frac{1}{s^{\alpha+1}} + \frac{1}{s^2}k_1 + \frac{1}{s^3}k_2 + \frac{1}{s^4}k_3. \tag{9}$$

To return to the problem in the variable x, apply the inverse Laplace transform in Equation (9) and obtain,

$$v(x) = -\frac{\overline{A}}{\Gamma(\alpha+1)}\ x^{\alpha} + k_1\ x + \frac{k_2}{2}\ x^2 + \frac{k_3}{6}\ x^3. \tag{10}$$

To determine the constants k_1, k_2 and k_3 the other boundary conditions are used $v(L) = 0$, $EIv''(0) = M(0) = 0$ e $EIv''(L) = M(L) = 0$.By doing this, we obtain

$$v(x) = -\frac{\overline{A}}{\Gamma(\alpha+1)}\ x^{\alpha} + k_1\ x + \frac{k_3}{6}\ x^3, \tag{11}$$

with $k_1 = \overline{A}L^{\alpha-1}\left[\frac{1}{\Gamma(\alpha+1)} - \frac{1}{6\Gamma(\alpha-1)}\right]$ and $k_3 = \overline{A}\frac{L^{\alpha-3}}{\Gamma(\alpha-1)}$, whose unit of $v(x)$ in this case is the meter.

Therefore, when $\alpha = 4$, there is A equals A barra and the solution converges to the solution of the classical model, a fundamental fact for the veracity of the fractional solution, according to [10].

For the case in which α is defined in the interval $4 < \alpha \leq 5$, the procedure is analogous to employed one for the previous case is obtained a solution in the neighborhood of 4 converging to Eq.(11). For more details on this development, see [3].

2.2 The Timoshenko-Ehrenfest theory

According to [3], the Timoshenko-Ehrenfest theory (TE) is modeled by the system of differential equations

$$\begin{cases} \dfrac{d^3\theta}{dx^3} = \dfrac{q}{EI} \\ \dfrac{dv}{dx} = \theta - \dfrac{EIc}{GA}\dfrac{d^2\theta}{dx^2} \end{cases} \tag{12}$$

where the dependent variables of the $\theta = \theta(x)$ and $v = v(x)$ system, represent the flexion rotation and the deflection of the barycentric axis, considering the effects of flexion and shear. The solution of the System (12) for the case of a simply supported beam with distributed load is presented by [3], whose result for the deflection is

$$v(x) = \frac{q}{EI}\left(-\frac{x^4}{24} + \frac{Lx^3}{12} - \frac{L^3x}{24}\right) + \frac{cq}{GA}\left(\frac{x^2}{2} - \frac{Lx}{2}\right). \tag{13}$$

It is easy to see that such a solution is a linear combination of bending deflection and shear deflection.

3 Applications and results

In this section, the scaled fractional FEB solution presented in Section (2) is used to evaluate the effects of shear and stress concentration at supports on the deflection of thick beams, as presented in Subsections (3.1) and (3.2). To this end, a prismatic, homogeneous, isotropic, linear elastic, simply supported beam subjected to an uniformly distributed load was implemented in ANSYS. The beam has rectangular cross section and is identical to the one used the developing of the analytical model as described at the beginning of Section (2).

To create an appropriate mesh, the element chosen was the 8 knot hexahedron, known in the literature as a continuous solid or 186 solid. As for the evaluation of mesh convergence for the beams implemented in ANSYS, iterations were carried out with mesh refinement, taking the normal maximum tensile stresses, aiming to determine the maximum possible size of the edge of the element, so that there is good convergence. It was found that in elements with edges smaller than or equal to $0.05m$, the tensions become constant, that is, the mesh converges.

The evaluation of the effects in each case studied is done by adjusting the solutions for deflections obtained in the ANSYS numerical solution with the solution obtained for the FEB analytical model implemented in MATLAB. The adjustment is made by varying the parameter α of the FEB solution for each fixed value K.

3.1 Evaluating the effects of shear

To evaluate and identify the shear effects, we chose the beam whose dimensions are $(b, h, L) = (0.3m; 1m; 5m)$, the same one used by [3] to study these effects, using the dimensionless FEB solution. Such a beam is suitable for the study, as it has an aspect ratio $\left(\frac{L}{h} = 5\right)$, which according to [8] and [9] are beams that present effects very strong shear. For the analysis of the evaluation and detection of shear effects, the scaled FEB and TE solutions were implemented in MATLAB. The Table (1) presents the values of the α parameter that identify the effects for each fixed value of K. In this case, K was interpreted as the rate of change in the length of the beam when subjected to loading. Another fact to be highlighted is that the adjustment between the curves was made by comparing the respective maximum deflections FEB and TE.

Table 1: Detection of shear effects on the beam $(b, h, L) = (0, 3m; 1m; 5m)$.

Constant K	Parameter α
$0.1L$	$\alpha = 4.04353$
$0.01L$	$\alpha = 4.02033$
$0.001L$	$\alpha = 4.01328$
$0.0001L$	$\alpha = 4.009864$

It can be seen from the Table (1) that when decreasing the value of the constant K, the corresponding α that detects the effect approaches the entire order 4. Therefore, the constant K can be interpreted as a control device to accommodate the α that contemplates the effect in the neighborhood of 4, where the solution is valid. This fact is relevant, as it makes the solution a tool capable of detecting much more intense effects with less memory effect. In Figure 1, there is a representation of the results for the case $K = 0.0001L$ and $\alpha = 4.009864$ in which, in (a), there is a comparison between the deflections and, in (b), the relative error between them.

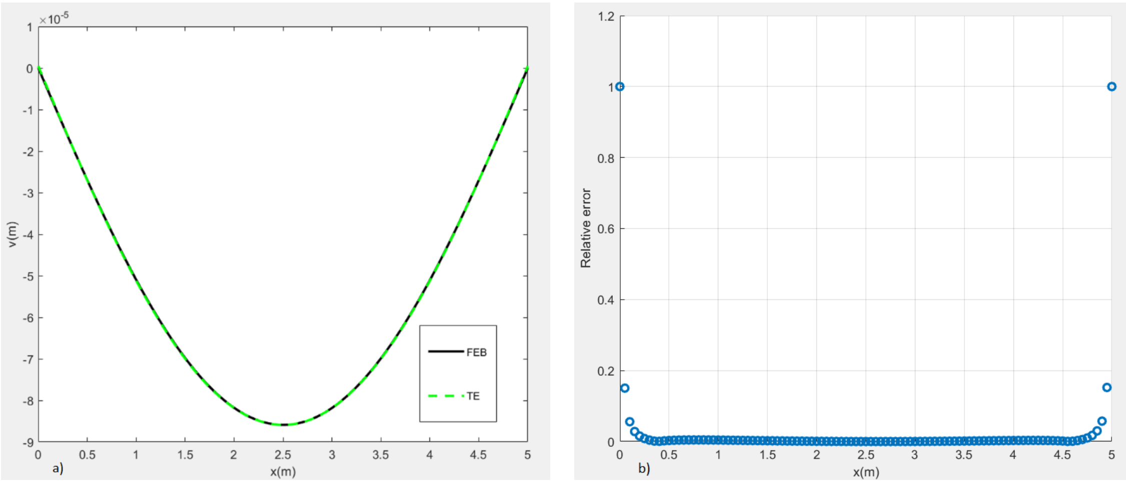

Figure 1: a) Graphics FEB and TE. b) Relative error. Source: From the authors.

3.2 Evaluating the effects of supports

To evaluate the effects of stress concentration in the supports on the deflection of a beam, a case of a thick beam is chosen for analysis, as types of beams in these such effects are more evident [11]. Such stress concentrations can cause variations in deflection even in cases where the beams are subjected to low stresses [2]. The purpose of the analysis is to identify the variation in transverse deflection caused by the stress concentration in the supports, using the variation of the parameter α incorporated in the FEB solution with dimension.

To analyze this case, the beam $(b, h, L) = (0.3m; 0.8m; 5m)$ was implemented in ANSYS and the linear simulation was carried out with load $q = 1 \times 10^4 N/m$. In this case, the transverse deflection obtained from ANSYS, taking the barycentric axis as its path, will have a contribution from the effects of moments, shear and stress concentration in the supports. Implementing the TE solution for the same beam in MATLAB and comparing the solution obtained from ANSYS with the TE solution obtained from MATLAB, it is unequivocal that the deflection obtained from ANSYS is slightly greater. This fact is related to the contribution to the deflection of the effects of stress concentration on the supports, as the analysis carried out in ANSYS was linear.

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To evaluate the effects of support, using the α parameter incorporated into the EB solution, we proceeded as follows: initially, the TE and FEB solutions were obtained for $\alpha = 4$, one was subtracted from the other, thus obtaining the residues related to the shear effects. Then, such residues were subtracted from the solution obtained in ANSYS. Finally, by adjusting the FEB solution scaled to the ANSYS solution, using the same procedure adopted in Subsection (3.1), we obtain the parameter α that evaluates the effects of supports on deflection. The Table (2) presents the results.

Table 2: Detection of the effects of supports on the beam $(b, h, L) = (0.3m; 0.8m; 5m)$.

Constant K	Parameter α
$0.1L$	$\alpha = 4.0499$
$0.01L$	$\alpha = 4.02325$
$0.001L$	$\alpha = 4.01519$
$0.0001L$	$\alpha = 4.01128$

Analyzing the results obtained in Table (2), it becomes clear once again that reducing the constant K implies accommodating the parameter α increasingly closer to 4. As already mentioned, this fact is relevant, as it makes the FEB solution capable of evaluating and identifying much more intense effects. In Figure 2, there is a representation of the results for the case $K = 0.0001L$ and $\alpha = 4.00660$ in which, in (a), there is a comparison between the arrows and, in (b), the relative error between them.

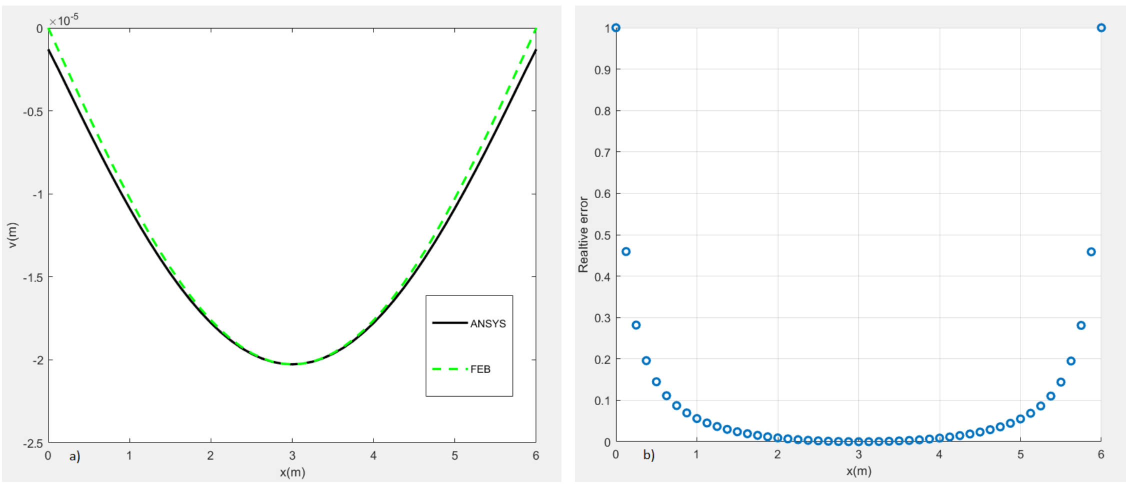

Figure 2: a) Graphics FEB and TE. b) Relative error. Source: From the authors.

4 Conclusions

In this work, a fractional model with dimensional analysis was proposed for the Euler-Bernoulli transverse deflection. The respective fractional solution was obtained considering the case of a simply supported beam subjected to an uniformly distributed static load. The model obtained was used to evaluate and detect the effects of shear, as well as the effects on deflection and stress concentration due to the presence of supports. The results obtained in the analysis of the two cases mentioned above showed that the model presented the expected behavior, which indicates it as a potential relevant tool for evaluating the detection of such effects. Given the positive perspective

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achieved in this study, we intend to present in future works regarding the evaluation of other effects, such as, for example, large deflections, buckling, plastics and spatial temperature variation.

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