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Long-time asymptotics of a linear higher-order water wave model

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This work deals with a class of higher-order Benjamin-Bona-Mahony (BBM) type equations:

$$\eta_t + \eta_x - \gamma_1 \eta_{xxt} + \gamma_2 \eta_{xxx} + \delta_1 \eta_{xxxxt} + \frac{3}{4}(\eta^2)_x + \gamma(\eta^2)_{xxx} - \frac{7}{48}(\eta_x^2)_x - \frac{1}{8}(\eta^3)_x = 0. \tag{1}$$

Introducing appropriate damping mechanisms for the associated linear model of the equation (1), we study the asymptotic behavior in time of the corresponding damped models. This is done both in the case of internal and boundary damping. We first address the internal stabilization problem: consider a periodic domain and introduce a localized damping mechanism acting in the equation. More precisely, we study the system

$$\begin{cases} \eta_t + \eta_x - \gamma_1 \eta_{txx} + \gamma_2 \eta_{xxx} + \delta_1 \eta_{txxxx} + \mathcal{B}\eta = 0 & \text{for } x \in (0, 2\pi),\ \ t > 0 \\ \partial_x^r \eta(t, 0) = \partial_x^r \eta(t, 2\pi), & \text{for } t > 0, \quad 0 \le r \le 3, \\ \eta(0, x) = \eta_0(x) & \text{for } x \in (0, 2\pi), \end{cases} \tag{2}$$

where r is an integer number, $\gamma_1,\ \delta_1 > 0$ and $\mathcal{B} : H_p^s(0, 2\pi) \longrightarrow H_p^s(0, 2\pi)$ is a bounded linear operator and $H_p^s(0, 2\pi)$ denotes the Sobolev space of 2π-periodic functions. Let $a \in C_p^\infty(0, 2\pi)$ a nonnegative function on $(0, 2\pi)$ with $a(x) > 0$ on a given open set $\Omega \subset (0, 2\pi)$.

We analyze the following cases for the operator $\mathcal{B}$: $\mathcal{B}\varphi = a(x)\varphi$ and $\mathcal{B}\varphi = -(a(x)\varphi_x)_x$. In both cases, the energy associated to (2) is given by

$$E(t) = \frac{1}{2}\int_0^{2\pi} (|\eta(t)|^2 + \gamma_1 |\eta_x(t)|^2 + \delta_1 |\eta_{xx}(t)|^2) dx \tag{3}$$

and (at least formally) it follows that $\frac{dE(t)}{dt} = -\int_\Omega \mathcal{B}\eta(t)\eta(t)dx$. Indeed, to obtain this last equality we multiply the equation in (2) by η and integrate by parts over $(0, 2\pi)$. Thus, since $\int_\Omega \mathcal{B}\eta(t)\eta(t)dx > 0$, the energy $E(t)$ decreases along the trajectories, and the operators $\mathcal{B}$ play the role of a damping mechanism. Next, we address the boundary stabilization problem. More precisely, we consider (2) with $\mathcal{B} \equiv 0$ on $(0, L)$, with initial condition $\eta(0, x) = \eta_0(x)$ and the following boundaries conditions

$$\begin{cases} \gamma_1 \eta_{xt}(t, 0) - \delta_1 \eta_{txxx}(t, 0) = \frac{3}{2}\eta(t, 0) + \gamma_2 \eta_{xx}(t, 0) & \text{for } \ t \ge 0, \\ \gamma_1 \eta_{xt}(t, L) - \delta_1 \eta_{txxx}(t, L) = -\frac{\eta(t, L)}{2} + \gamma_2 \eta_{xx}(t, L) & \text{for } \ t \ge 0, \\ \delta_1 \eta_{txx}(t, 0) = \left(1 - \frac{\gamma_2}{2}\right) \eta_x(t, 0) & \text{for } \ t \ge 0, \\ \delta_1 \eta_{txx}(t, L) = -\left(1 + \frac{\gamma_2}{2}\right) \eta_x(t, L) & \text{for } \ t \ge 0. \end{cases} \tag{4}$$

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In this case, the energy associated to this model is given by

$$E(t) = \frac{1}{2}\int_0^{2\pi} (|\eta(t)|^2 + \gamma_1|\eta_x(t)|^2 + \delta_1|\eta_{xx}(t)|^2)dx \tag{5}$$

and (at least formally) it follows that $\frac{dE(t)}{dt} = -(|\eta(t,L)|^2 + |\eta(t,0)|^2) - (|\eta_x(t,L)|^2 + |\eta_x(t,0)|^2)$. Thus, the energy $E(t)$ is decreasing and the boundary conditions play the role of a feedback-damping mechanism. We show that some of Rosier's results [3] can be extended to the dissipative models introduced above. Roughly speaking, once the unique continuation property and the well-posedness of solutions have been proved, our results on the asymptotic behavior of the solutions of (2) and (2)-(4) can be described as follows:

Theorem 1. *For any $\eta_0 \in H_p^2(0,2\pi)$, the solution η of (2) satisfies*

$$\begin{aligned} &\eta(t) \to 0 \quad \textit{weakly in } H_p^2(0,2\pi),\\ &\eta(t) \to 0 \quad \textit{strongly in } H_p^s(0,2\pi), \quad \textit{for all } s \in [0,2), \ \textit{as } t \to \infty \end{aligned}$$

If the damping mechanism $\mathcal{B}\varphi$ involves one derivative, the solutions converge to the mean of the initial datum.

Theorem 2. *For any $\eta_0 \in H_p^2(0,2\pi)$, the solution η of (2) satisfies*

$$\begin{aligned} &\eta(t) \to [\eta_0] \quad \textit{weakly in } H_p^2(0,2\pi),\\ &\eta(t) \to [\eta_0] \quad \textit{strongly in } H_p^s(0,2\pi), \quad \textit{for all } s \in [0,2), \quad \textit{as } t \to \infty, \ \textit{where } \ [f] := \frac{1}{2\pi}\int_0^{2\pi} f(x)dx. \end{aligned}$$

Finally, when the equation is dissipated through the boundary conditions, we obtain the following result:

Theorem 3. *For any $\eta_0 \in H^2(0,L)$, the solution η of (2)-(4) with $\mathcal{B} \equiv 0$ and $\gamma_2 = 0$ satisfies*

$$\begin{aligned} &\eta(t) \to 0 \quad \textit{weakly in } \ H^2(0,L),\\ &\eta(t) \to 0 \quad \textit{strongly in } \ H^s(0,L), \quad \textit{for all } \ s \in [3/2,2), \ \textit{as } t \to \infty. \end{aligned}$$

Our proofs rely on the approach developed in [3] to study similar problems for the scalar BBM equation and [1, 2] for the BBM-type systems.

Referências

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