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Momentum Operators on Continuous Markov Evolution Algebras

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Abstract

In this work we introduce the notion of momentum operator on a family of evolution algebras indexed by a time-parameter $t \geq 0$. Also, we study its spectra in the case of finte-dimensional evolution algebra. Thus, this work is naturally divided into two parts. In the first part we give the main definitions on (continuous-time) Markov evolution algebras and we present some basic results on these algebras. For more details on continuous evolution algebras see [6, 7].

In the second part, we introduce the notion of momentum operator on such structures. In [5] the author study these operator on finite graphs. Then we proceed to determine its spectra in the context of continuous-time Markov evolution algebras.

Introduction

Evolution algebras were introduced by Tian and Vojtechovsky (see [8]). A special class of such algebras called continuous evolution algebras and their connection to continuous-time Markov processes are study in [9]. Its notion has been recently revisited in [6] where this problem is formulated in terms of differentiable matrix valued functions.

Evolution algebras are nonassociative algebras admitting natural bases for which the only non-vanishing products arise from the squares of the natural basis elements. A real Markov evolution algebra arise when its natural basis comes from a nonnegative row stochastic (i.e. Markov) structure matrix. Continuous-time Markov evolution algebras were redefined in terms of stochastic semigroups.

Given a finite dimensional (real) vector space $\mathcal{E}$ with basis $\mathcal{B} = \{e_1, \ldots, e_n\}$, a family $\mathcal{E}(t) = \{\mathcal{E}_t = (\mathcal{E}, m(t))\}_{t \geq 0}$ of evolution algebras with multiplication

$$m(t)(e_i \otimes e_j) = e_i \cdot_t e_j = \begin{cases} \sum_{k=1}^{n} a_{ik}(t) e_k, & i = j = 1, \ldots, n; \\ 0, & \text{otherwise}; \end{cases}$$

is a continuous time Markov evolution algebra (CT-Markov EA) if the structure matrices $\{\mathbf{A}(t)\}_{t \geq 0}$ (of each $\mathcal{E}_t$ w.r.t. $\mathcal{B}$) define a standard stochastic semigroup on the finite index set $\Lambda = \{1, \ldots, n\}$. Then, for each $t, s \geq 0$:

(i) $\mathbf{A}(t)$ is a Markov matrix.

(ii) $\mathbf{A}(0) = \mathbf{I}_n$.

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(iii) $\mathbf{A}(t+s) = \mathbf{A}(t)\mathbf{A}(s)$ (Chapman-Kolmogorov equation or semigroup property).

(iv) $\lim_{t\to 0^+} \mathbf{A}(t) = \mathbf{A}(0) = \mathbf{I}_n$ componentwise (standard property).

Finite state standard stochastic semigroups are solutions of Backward and Forward Kolmogorov differential equations

$$\mathbf{A}'(t) = \mathbf{Q}\mathbf{A}(t)$$

and

$$\mathbf{A}'(t) = \mathbf{A}(t)\mathbf{Q}.$$

with initial condition $\mathbf{A}(0) = \mathbf{I}_n$. The unique solution is $\mathbf{A}(t) = e^{t\mathbf{Q}}$, where $\mathbf{Q}$ is a rate matrix of a continuous-time Markov chain. It also holds $\mathbf{A}'(0) = \mathbf{Q}$. Moreover, since $\det(\mathbf{A}(t)) = e^{tr(t\mathbf{Q})}$ we may conclude that matrices in finite standard stochastic semigroups are nonsingular matrices belonging to the stochastic group $S(n, \mathbb{R})$.

After these preliminaries we will discuss under which conditions we can define a discrete version of a momentum operator.

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