Proceeding Series of the Brazilian Society of Computational and Applied Mathematics, v. 10, n. 1, 2023.

Trabalho apresentado no XLII CNMAC, Universidade Federal de Mato Grosso do Sul - Bonito - MS, 2023

Proceeding Series of the Brazilian Society of Computational and Applied Mathematics

Symmetric Measures and Associated Orthogonal Polynomials Obtained from Modified Positive Chain Sequences

Daniel O. Veronese[1]
UFTM-ICTE, Uberaba, MG

Given a nontrivial positive measure φ (i.e., a Borel measure with infinite support), we say that $\{P_n(x)\}_{n=0}^{\infty}$ is the sequence of monic orthogonal polynomials on the real line (MOPRL, in short) associated with φ, if $P_n(x)$ is a monic polynomial of degree n satisfying

$$\int_{\mathbb{R}} x^j P_n(x) d\varphi(x) = \delta_{n,j}\rho_n, \quad 0 \le j \le n, \tag{1}$$

where $\delta_{n,j}$ is the Kronecker delta and $\rho_n \neq 0$ for $n \ge 0$.

It is well known (see [1] and [2]) that the polynomials $P_n(x)$, $n \ge 0$, satisfy the three-term recurrence relation

$$P_{n+1}(x) = (x - \beta_{n+1}^{\varphi})P_n(x) - \alpha_{n+1}^{\varphi} P_{n-1}(x), \quad n \ge 1, \tag{2}$$

with $P_0(x) = 1$ and $P_1(x) = x - \beta_1^{\varphi}$. Moreover, the coefficients in (2) can be given as follows

$$\beta_n^{\varphi} = \frac{\int_{\mathbb{R}} x P_{n-1}^2(x) d\varphi(x)}{\int_{\mathbb{R}} P_{n-1}^2(x) d\varphi(x)} \quad \text{and} \quad \alpha_{n+1}^{\varphi} = \frac{\int_{\mathbb{R}} P_n^2(x) d\varphi(x)}{\int_{\mathbb{R}} P_{n-1}^2(x) d\varphi(x)}, \quad n \ge 1. \tag{3}$$

We say that the nontrivial positive measure φ is symmetric when the following identity holds

$$d\varphi(x) = -d\varphi(-x), \quad x \in \mathbb{R}. \tag{4}$$

In this case, it is well known that the coefficients $\beta_n^{\varphi}, n \ge 1$, are equal to zero, and therefore, the MOPRL associated with φ only depends on α_{n+1}^{φ}, $n \ge 1$. Another well known result is that, when $supp(\varphi) \subset [-1, 1]$, the sequence $\{\alpha_{n+1}^{\varphi}\}_{n=0}^{\infty}$ is a positive chain sequence.

Following [3], a sequence $\{a_n\}_{n=1}^{\infty}$ is a positive chain sequence if there exists another sequence, say $\{g_n\}_{n=0}^{\infty}$, with $0 \le g_0 < 1$, and $0 < g_n < 1, n \ge 1$, such that

$$a_n = (1 - g_{n-1})g_n, \quad n \ge 1.$$

The sequence $\{g_n\}_{n=0}^{\infty}$ is called a parameter sequence for $\{a_n\}_{n=1}^{\infty}$. The minimal parameter sequence $\{m_n\}_{n=0}^{\infty}$ for $\{a_n\}_{n=1}^{\infty}$ is the one that satisfy $m_0 = 0$. On the other hand, the maximal parameter sequence $\{M_n\}_{n=0}^{\infty}$ is the one that satisfy $g_n \le M_n, n \ge 0$, for any parameter sequence $\{g_n\}_{n=0}^{\infty}$.

In the last years, positive chain sequences have been also applied to the study of orthogonal polynomials on the unit circle and corresponding nontrivial measures. As an example, we can cite [4] and [5].

[1]daniel.veronese@uftm.edu.br

In the present study, we initially consider a symmetric nontrivial positive measure for which the associated sequence $\{\alpha_{n+1}^{\varphi}\}_{n=0}^{\infty}$ that appears in (2) is a nonsingle positive parameter chain sequence. It means that its minimal parameter sequence differs from its maximal parameter sequence.

Then, the maximal parameter sequence $\{M_n^{\varphi}\}_{n=0}^{\infty}$ for $\{\alpha_{n+1}^{\varphi}\}_{n=0}^{\infty}$ is such that

$$\alpha_{n+1}^{\varphi} = (1 - M_{n-1}^{\varphi})M_n^{\varphi}, \quad n \geq 1,$$

with $M_0^{\varphi} > 0$.

In this work we analyze the behaviour of the nontrivial positive measure $\hat{\varphi}$ for which the associated orthogonal polynomials $\{\hat{P}_n(x)\}_{n=0}^{\infty}$ are such that the coefficients in the three-term recurrence relation satisfy

$$\alpha_{n+1}^{\hat{\varphi}} = (1 - g_{n-1}^{\hat{\varphi}})g_n^{\hat{\varphi}}, \quad n \geq 1, \tag{5}$$

with

$$g_{2n}^{\hat{\varphi}} = 1 - M_{2n}^{\varphi} \quad \text{and} \quad g_{2n+1}^{\hat{\varphi}} = M_{2n+1}^{\varphi}, \quad n \geq 0. \tag{6}$$

We show that $\hat{\varphi}$ and φ are connected by the following identity

$$d\hat{\varphi}(x) = \left(\frac{x^2 - 1}{x|x|}\right) d\varphi(\sqrt{1 - x^2}), \quad x \in [-1, 0) \cup (0, 1]. \tag{7}$$

Moreover, it is shown that the mass of $\hat{\varphi}$ at $x = 0$ is equal to zero. Thus, if the measure φ is known, the relation (7) provides the complete information regarding $\hat{\varphi}$.

As a consequence, from (5), (6) and (7), we get a procedure to generate new examples of symmetric orthogonal polynomials on the real line with coefficients explicitly given.

Finally, we provide conditions in order to guarantee that the new parameter sequence $\{g_n^{\hat{\varphi}}\}_{n=0}^{\infty}$ (generated from $\{M_n^{\varphi}\}_{n=0}^{\infty}$) will also be the maximal parameter sequence for $\{\alpha_{n+1}^{\hat{\varphi}}\}_{n=0}^{\infty}$. Namely, it is possible to show that $\{g_n^{\hat{\varphi}}\}_{n=0}^{\infty}$ is the maximal parameter sequence for $\{\alpha_{n+1}^{\hat{\varphi}}\}_{n=0}^{\infty}$ if and only if φ has no mass at $x = 0$.

References

[1] T. S. Chihara. **An Introduction to Orthogonal Polynomials**. Mathematics and its Applications Series. Gordon and Breach, 1978. ISBN: 978-0-677-04150-6.

[2] M. E. H. Ismail. **Classical and Quantum Orthogonal Polynomials in One Variable**. Encyclopedia of Mathematics and its Application. Cambridge Univ. Press, 2005. ISBN: 978-0-521-78201-2.

[3] T. S. Chihara. "Chain Sequences and Orthogonal Polynomials". In: **Transactions of the American Mathematical Society** 104 (1962), pp. 1–16. DOI: `10.2307/1993928`.

[4] M. S. Costa, H. M. Felix, and A. Sri Ranga. "Orthogonal polynomials on the unit circle and chain sequences". In: **Journal of Approximation Theory** 173 (2013), pp. 1–20. DOI: `10.1016/j.jat.2013.04.009`.

[5] D. O. Veronese. "Orthogonal Polynomials and Quadrature Rules on the Unit Circle Associated with Perturbations of Symmetric Measures". In: **Journal of Computational and Applied Mathematics** 375 (2020), pp. 1–20. DOI: `10.1016/j.cam.2020.112808`.

