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Gradient Pattern Analysis of Coupled Map Lattices: Insights into Transient and Long-Term Behaviors

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Abstract. Gradient Pattern Analysis (GPA) is a useful technique for analyzing the dynamics of nonlinear 2D-spatiotemporal systems, which is based on the gradient symmetry-breaking properties of a matrix snapshot sequence. GPA has found numerous applications in dynamic systems, particularly in studying logistic Coupled Map Lattices (CMLs) and Swift-Hohenberg amplitude equations. In this work, we propose a new mathematical operation related to the first gradient moment (G_1) defined by the GPA theory. The performance of this new measure is evaluated by applying it to two chaotic CML models (Logistic and Shobu-Ose-Mori). The GPA using the new parameter (G_1) provides a more accurate analysis, allowing the identification of conditions that partially break the gradient symmetry over time. Based on the GPA measurements (G_1, G_2 and G_3), including a combined analysis with the chaotic parameters, our results demonstrate the potential to analyze chaotic spatiotemporal systems improving our understanding of their underlying dynamics.

Keywords. Coupled Map Lattice (CML), Gradient Pattern Analysis (GPA), Chaotic systems.

1 Introduction

Coupled Map Lattices (CMLs) act as a mathematical model to study the behavior of non-linear dynamical systems that may exhibit chaotic behavior. Different CML models (1D and 2D), in relation to the master equation, have been employed across diverse areas, including physics [2, 13], social sciences [11], and cryptography for generating pseudo-random numbers [3]. These systems exhibit complex dynamics, such as oscillatory relaxation [4, 8], kink, anti-kink, and synchronization [2], whose understanding can have important practical implications. A key feature that can alter the local instability that defines pattern formation is the symmetry imposed by the initial condition, boundary conditions and the nonlinearity of the model. The Gradient Pattern Analysis (GPA) formalism allows investigation of the formation of non-linear patterns directly in the gradient lattice of the system (e.g., [1, 6]). The main advantage of this approach is to understand the formation of spatiotemporal patterns from the small phase and modulus fluctuations that occur in the gradient lattice of a CML. In this work, based on the improved application of the GPA, described in section 2, on the CMLs presented in section 3, we present a parameter space analysis that contributes to strengthening the conclusions presented in the last section.

2 Gradient Pattern Analysis

Gradient Pattern Analysis (GPA)[4] is a mathematical-computational formalism proposed to characterize two-dimensional spatial patterns from the gradient lattice of an amplitude matrix.

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In this technique, the Gradient Lattice (GL) is computed in the x-direction and the y-direction, whenever possible, with Central Finite Difference. The boundaries are measured with forward and backward finite differences.

An important feature of describing underlying physical processes is symmetry, which is an indicator of balance and stability. This feature has been generalized to GLs in GPA formalism, where the vector lattice is segmented into symmetrical, asymmetrical, and homogeneous regions. The Asymmetric Gradient Method (AGM) is a technique to detect symmetry breaking, which had major importance in recent works[1, 6]. In this method, vectors equidistant to the matrix center are compared with each other. The first step is to determine homogeneous regions, where the modulus is smaller than ξ. Then, symmetry and asymmetry are sought in the remaining regions. If there is at least one vector where the vectorial sum produces a vector with a size smaller than ξ, then it is classified as symmetrical, otherwise, they are asymmetrical.

In this way, the mathematical representation of gradients can assume different designs that highlight intrinsic information of the two-dimensional partial derivatives. In order to organize the metrics, Rosa et al. [7] proposes the classification of those metrics with a similar methodology to classical physics moments and statistical moments to the Gradient Lattice represented in Figure 1, known as *Gradient Moment* (GM).

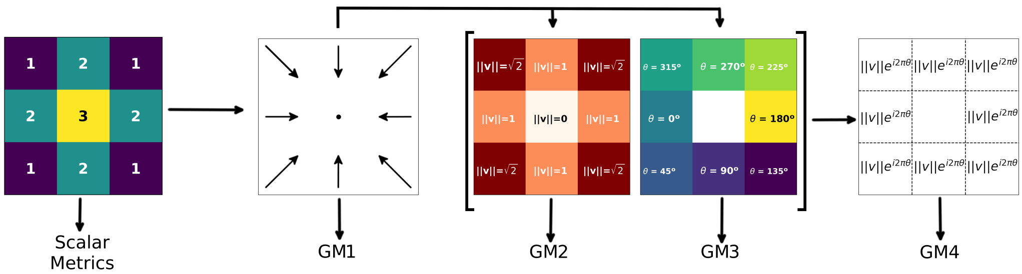

Figure 1: Scheme representing the GPA operations and respective Gradient Moments (GM). Filtering the information from matrices to compose a new numerical lattice can enhance or diminish aspects of the observed pattern. Thus, every metric extracted from a given numerical lattice is a GM in GPA formalism. The spatial variances form the Gradient Lattice (GL): which is the first GPA output. The modulus and the phases from the GL are the second and third numerical lattices. Finally, the complex notation of the GL composes the fourth Lattice.

2.1 The Gradient Moments

The GM1 combines spatial GL information of the position, modulus, and phases. A methodology to combine that information is the Delaunay triangulation, where every vector of the GL is added to its position, and then the set of resulting points is triangulated [5–7]. The triangulation criterion was established to maximize the space occupied with the smallest possible number of triangles since the Delaunay triangulation maximizes the internal angles. With the triangulation set, several features can be extracted, such as the classical metric $G_1^C = (N_C - N_V)/N_V$, where which G_1^C is a relation between the number of Delaunay connections N_C and the number of vectors N_V. This metric is bounded to the interval $[0, 2[$, as the pattern becomes more intricate the metric increases, a result that can be observed by measuring G_1^C in random matrices [5]. In this work, we explore a new metric for $GM1$, which is a relation between the distribution of Delaunay connection size. Given the set of connection lengths that are larger than the median connection lengths L_{sup}, and the set of connection lengths that are smaller than the median connection lengths L_{inf}, G_1 is

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measured as:

$$G_1 = \frac{\overline{L}_{\text{sup}} - \overline{L}_{\text{inf}}}{\max(L)}, \tag{1}$$

that is a relation between the L_{sup} average and the L_{inf} average normalized by the largest Delaunay connection length $\max(L)$. If all triangles are equal $L_{\text{sup}} = L_{\text{inf}}$, and thus $G_1 = 0$. On the other hand, a sparse pattern has a large variation of triangle connections, in an extreme scenario, $L_{\text{inf}} \approx 0$ and $L_{\text{inf}} \approx \max(L)$, in which $G_1 \approx 1$.

In a more refined approach, the extraction of features from vector modules is sought. A procedure adopted in $GM2$ is:

$$G_2 = \frac{V_A}{V}\left(1 - \frac{\left|\sum_{i=0}^{V_A} v_i\right|}{2\sum_{i=0}^{V_A} |v_i|}\right), \tag{2}$$

which describes the relation between every vector modulus $|v_i|$ and the modulus of the overall vector $|\sum v_i|$, in a lattice with V vectors and V_A asymmetrical vectors. When all vectors in the lattice are misaligned, $|\sum v_i| = 0$, and therefore $G_2 = V_A/V$. On the other hand, if all vectors are aligned, for instance, a 'gradient image', then $|\sum v_i| = k\,|v_i| = \sum |v_i|$ and $G_2 = V_A/(2V)$. Thus, the term $|\sum v_i| / \sum |v_i|$ is a gradient alignment term.

Another approach to pattern characterization is phase metrics, which belong to $GM3$. Although the $GM3$ has been formally defined [7], until the present work, no metric has been proposed for a practical application. We introduce the metric:

$$G_3 = \frac{V_A}{2V} + \sum_{i=0}^{V_A} \frac{\cos(\theta_i)\cos(\phi_i) + \sin(\theta_i)\sin(\phi_i) + 1}{4V_A}, \tag{3}$$

that is a relation between each phase θ_i and the corresponding direction of the phase location ϕ_i. In a scenario where all vectors of the lattice are aligned outwards, θ_i and ϕ_i have the same direction, which implies in $\cos(\theta_i)\cos(\phi_i) + \sin(\theta_i)\sin(\phi_i) = 0$, and therefore $G_3 = (2V_A + V)/4V$.

To summarize the gradient moments we can describe the information about the vector's location, modulus, and phase that compose the first Gradient Moment ($GM1$), the second Gradient Moment ($GM2$) is related to the gradient modulus, and the third Gradient Moment ($GM3$) is related to the phases of the gradient lattice and the direction of each individual vector. To validate and understand the presented technique, we define the CML as a canonical application of the technique, which is presented in the following section.

3 Coupled Map Lattices

Spatially related chaotic systems can show a wide variety of dynamical behaviors, such as spatial synchronization, frozen chaos, and spatially driven chaos depending on the parametrization and initial condition [2, 10, 12]. The spatial term in those systems can induce stability in unstable local dynamics, or vice-versa. Many of these systems have been proposed in the literature to describe natural processes, where the regime is known as the Edge of Chaos [10]. Here we present the Coupled Map Lattice (CML)[5], a dynamical system that has been proposed as an extension to classical chaotic maps.

The CML is composed of oscillators $f(u^t_{x,y})$ in a regular grid, in which spatial interaction is controlled by a coupling parameter ϵ. The system state $u^t_{x,y}$ is updated at every time step according to:

[5] The source code is public-available at: https://github.com/rsautter/CML-CoupledMapLattice

$$u^{t+1}_{x,y} = (1-\epsilon)f(u^t_{x,y}) + \frac{\epsilon}{4}\left(f(u^t_{x+1,y}) + f(u^t_{x-1,y}) + f(u^t_{x,y+1}) + f(u^t_{x,y-1})\right). \quad (4)$$

In CML studies, the logistic map has become a standard case [4, 12, 13], which has the form: $f(u) = \lambda u(1-u)$. The term λ is a parameter that controls the local regime of the series. In a range $\lambda = [3, 4]$, the map presents all 2^N periodic orbits and chaotic regimes.

Another interesting map to study is the so-called Shobu-Ose-Mori (SOM) map. It has been proposed to study intermittent turbulence [9], which is given by the following piece-wise linear map:

$$f(u) = \begin{cases} u + 0.2, & u < \gamma \\ \alpha(u-0.8)+1 & \gamma \le u < 0.8 \\ (1-u)/\beta & u \ge 0.8 \end{cases}, \quad (5)$$

where the parameter $\gamma = 1/(0.8+\alpha)$. The intermittency proposed in SOM map is related to the difference in linear dynamic for $u < \gamma$ and the bursting dynamic for $u \ge 0.8$. The map also presents an almost-unstable dynamic for the set of parameters: $\alpha = 0.6$, $\beta = 0.2$. Any small increase in the α parameter (i.e. 10^{-5}) leads to instability.

There are several manners to study dynamical systems, the following subsections concentrate on two approaches that are more common in the CML literature and the GPA applications. The former approach is the study of transient regimes, where a given metric is measured over time to generate a time series. The latter is a classification approach, where the parameters of the model are correlated with the metric.

3.1 Transient Regime Characterization

Predicting the exact dynamics of a system is often unfeasible, due to numerical-computational problems. An alternative to this problem is to determine the time series regime, where the system is characterized by a set of common characteristics at a given time range. Most GPA applications aim to characterize transient regimes in a matter of symmetry/asymmetry and GM [4, 5, 7]. Since the GM are all GMs are measured at every snapshot of the system, resulting in a time series, having some instance Figure 2. In these examples, the system begins in a symmetrical 2D Gaussian, which is transformed into more complex structures as the system iterates, until the symmetry breaks. Subsequent analyses are related to oscillations metrics, statistical metrics, or clustering processes, with the objective to determine the transient stages and series extremes.

The observed regimes have been divided into four categories, according to the nature of the transient stability; (i) stable regime, (ii) boundary-driven unstable regime, (iii) local chaos-driven unstable regime, and (iv) a mixing of boundary-driven and local chaos-driven unstable regime. The logistic CML and SOM CML have different symmetry-breaking processes. SOM CML and logistic CML follow a dynamic of type (i) in the beginning, where all GMs are zero. The symmetry-breaking process is different in both systems; the logistic CML reaches a type (ii) regime, whereas SOM CML reaches type (iii). Afterward, both systems reach a type (iv) regime, in which the metrics are relaxed.

3.2 Classification of Long-term Regimes

Ideally, the analysis of the dynamics of a system should be carried out during the transient to understand the processes involved and determine whether the system achieved relaxation. However, some observational data is limited in terms of history. Therefore, another common challenge is long-term regime characterization, such as galaxy morphology classification [1, 6].

The long-term characterization is also important to understand the effects of the parameter in CMLs. Once, the presence or absence of symmetry in the system is an indicator of the underlying

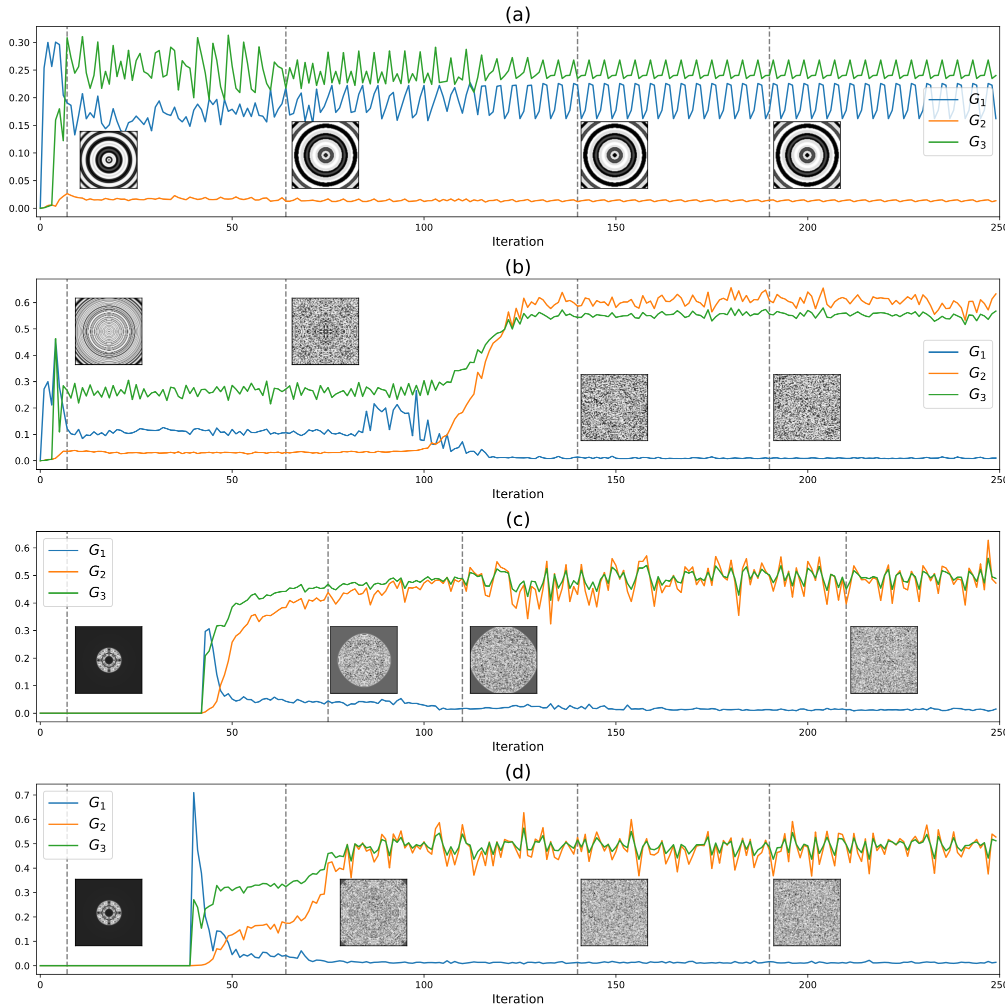

Figure 2: Gradient Moments of a 128x128 CML and some snapshots. (a) Logistic CML in locally stable conditions ($\lambda = 3.54$, $\epsilon = 0.3$). (b) Logistic CML in locally unstable conditions ($\lambda = 4.00$, $\epsilon = 0.3$). (c) SOM CML in locally stable conditions ($\alpha = 0.6$, $\epsilon = 0.3$). (d)SOM CML in locally unstable conditions ($\alpha = 0.6 + 10^{-5}$, $\epsilon = 0.3$).

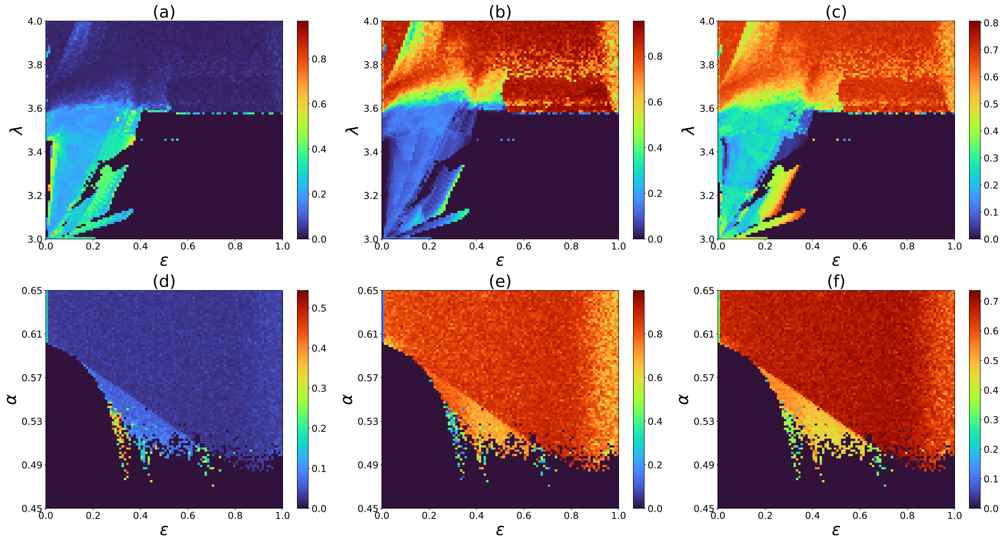

Figure 3: Parametric-spaces of 32x32 CMLs with Gaussian initial condition. For every parameter set, the system is iterated 1,000 times and the Gradient Moment is measured. The charts (a),(b), and (c) are Logistic CML. The plots (d), (e), and (f) are the SOM CML. (a) and (d) are G_1 parametric space, (b) and (e) are G_2 parametric space, (c) and (f) are G_3 parametric space.

physical processes. In this sense, we propose a long-term analysis of 32x32 logistic CML, which results are presented in Figure 3. In this analysis, the system starts with a 2D-Gaussian initial condition. Then, the system is iterated 1,000 times and the GMs are measured in the last snapshot to compose the so-called Parametric-space.

Without coupling ($\epsilon = 0.0$), all oscillators follow a dynamical behavior of a single map. Thus, for the parameter $\lambda < 3.57$ the system tends to a 2^N periodic orbit configuration, that preserves the system symmetry. Also, small numerical errors are magnified in an uncoupled chaotic regime ($\lambda > 3.57$), which breaks the symmetry in systems with symmetrical initial conditions. However, the coupling process can induce some instabilities or stability depending on the initial condition and parametrization. For instance, a parametrization ($\epsilon = 0.3$, $\lambda = 3.2$) has some localized instabilities that break the symmetry.

Concluding Remarks

In this paper, we have explored the dynamics of Coupled Map Lattices (CMLs) and their applications in the Gradient Pattern Analysis (GPA) context. Our results demonstrate that GPA is a powerful tool for understanding transient and long-term behaviors, which is important in a wide variety of applications, from physical and biological to social and economic systems. The analysis of symmetry-breaking in CMLs allows us to gain insights into the underlying processes that drive the evolution of these systems. Our work emphasizes the importance of characterizing the dynamics of a system during the transient phase, as this can provide valuable information and help determine whether the system has reached a state of relaxation. Additionally, we have shown the significance of long-term regime characterization for understanding the effects of parameters in CMLs, which may be useful for future studies on spatiotemporal chaos. Overall, our findings highlight the potential of GPA as a useful tool for the analysis of nonlinear spatiotemporal patterns.

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