Proceeding Series of the Brazilian Society of Computational and Applied Mathematics, v. 10, n. 1, 2023.

Trabalho apresentado no XLII CNMAC, Universidade Federal de Mato Grosso do Sul - Bonito - MS, 2023

Proceeding Series of the Brazilian Society of Computational and Applied Mathematics

Fractional Differentiation and Integration for Fuzzy Functions on Time Scales

Mina Shahidi,[1] Estevão Esmi,[2] Laécio Carvalho de Barros[3]
Department of Applied Mathematics, University of Campinas, Campinas, Brazil

Abstract. In this paper, we propose a new definition of the fractional derivative and fractional integral for fuzzy functions on time scales. The introduced derivative is a natural extension of the Hukuhara derivative. Furthermore, some properties of the introduced derivative and integral are studied. Some examples are provided to illustrate the obtained results.

Keywords. Time Scales, Fuzzy Functions, Fuzzy Fractional Derivative, Fuzzy Fractional Integral

1 Introduction

Fractional calculus, which leads to the examination and applications of derivatives and integrals of arbitrary order, has a long history [9]. In recent years, the theory has attracted a noticeable interest in various areas and so has been applied in diverse fields such as engineering, electrochemistry, biology, etc [4, 5].

The calculus of time scales, introduced by Stefan Hilger in 1990 [7], has become an important area of study in mathematics and has gained significant attention in recent years. This is not only because it can unify continuous and discrete calculus but also because time scale calculus has tremendous potential for applications such as population dynamics, economics, heat transfer, etc. The study of fractional derivatives on continuous, discrete, and more generally, a time scale is a well-known matter provided an excellent tool for modeling real problems in scientific fields and diffusion processes and so on.

However, in a real-world phenomenon, we face terms that are inherently vague or uncertain. For this reason, Zadeh [12] proposed fuzzy set theory, which provided an important theoretical basis for dealing with uncertain problems. Therefore, the basic concepts and definitions in the calculus of fuzzy functions on time scales are introduced by Fard and Bidgoli [6]. Vasavi et al. [11] studied Hukuhara delta derivative for fuzzy functions on time scales. Recently, in [10], Shahidi and Khastan introduced a fractional derivative based on the Hukuhara difference for fuzzy functions on time scales.

In this paper, using the Hukuhara difference [1] and the differentiability and integrability concepts proposed in [2], we introduce and study a new concept of fuzzy fractional derivative and integral on time scales. Therefore, this study extends the presented results in [2, 11].

The paper is organized as follows. In Section 2, we recall some basic concepts and results that will be used in the rest of the paper. In Section 3, we study and investigate some properties of fractional derivative and integral for fuzzy functions on time scales.

[1]shahidi@ime.unicamp.br
[2]eelaureano@gmail.com
[3]laeciocb@ime.unicamp.br

2 Preliminaries

In what follows, we present some definitions and introduce the required notions that will be used throughout the paper. A fuzzy set on $\mathbb{R}$ is a mapping $v : \mathbb{R} \to [0, 1]$. For $r \in (0, 1]$, the r-level set of v is given by $[v]_r = \{t \in \mathbb{R}|v(t) \geq r\}$ and for $r = 0$ by the closure of the support $[v]_0 = cl\{t \in \mathbb{R}|v(t) > 0\}$.The set of normal, fuzzy convex, upper semi-continuous, and compact supported fuzzy sets is called the space of fuzzy numbers and it is denoted by $\mathbb{R}_F$. The notation

$$[v]_r = [v_r^-, v_r^+],$$

denotes the r-level set of v. The addition in $\mathbb{R}_F$ and for $\lambda \in \mathbb{R}$, the scalar multiplication are given level set by

$$[v + w]_r = [v]_r + [w]_r,\ [\lambda v]_r = \lambda[v]_r,\ v, w \in \mathbb{R}_F,\ r \in [0, 1].$$

It is well-known [1] that $(\mathbb{R}_F, D)$ is a complete metric space with the metric structure $D : \mathbb{R}_F \times \mathbb{R}_F \to \mathbb{R}^+ \cup \{0\}$ given by the Hausdorff distance

$$D(v, w) = \sup_{r \in [0,1]} \max\{|v_r^- - w_r^-|, |v_r^+ - w_r^+|\},\ v, w \in \mathbb{R}_F.$$

The well-known properties of the metric D are given as follows:

- $D(v + z, w + z) = D(v, w), \forall v, w, z \in \mathbb{R}_F,$
- $D(\mu v, \mu w) = |\mu| D(v, w), \forall v, w \in \mathbb{R}_F, \mu \in \mathbb{R},$
- $D(v + w, z + e) \leq D(v, z) + D(w, e), \forall v, w, z, e \in \mathbb{R}_F,$
- $D(\lambda v, \mu v) = |\lambda - \mu| D(v, \tilde{0})$, for $\lambda\mu > 0$, where $\tilde{0} = \chi_{\{0\}}$.

Definition 2.1. *[1] Let $v, w \in \mathbb{R}_F$. The Hukuhara difference (H-difference, for short) is the fuzzy number $z \in \mathbb{R}_F$, provided that it exists, such that*

$$v \ominus w = z \Longleftrightarrow v = w + z.$$

Lemma 2.1. *[8] Let $u, v, w, z \in \mathbb{R}_F$ and $a, b \in \mathbb{R}$, then we have*

i. $au \ominus bu = (a - b)u, ab > 0.$

ii. If $u \ominus v$ exists and $\lambda \in \mathbb{R}$, then $\lambda u \ominus \lambda v$ exists and $\lambda(u \ominus v) = \lambda u \ominus \lambda v$.

iii. If $u \ominus w$ and $v \ominus z$ exist, then $(u + v) \ominus (w + z)$ exists and

$$(u + v) \ominus (w + z) = (u \ominus w) + (v \ominus z).$$

Definition 2.2. *[3] A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers.*

Definition 2.3. *[3] For $t \in \mathbb{T}$, the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$. In addition, the graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) := \sigma(t) - t$.*

Definition 2.4. *[3] If $\sigma(t) > t$, then the point t is called right-scattered. Furthermore, if $\rho(t) < t$, then t is left-scattered. If $\sigma(t) = t$, then t is right-dense and if $\rho(t) = t$, then t is left-dense.*

DOI: 10.5540/03.2023.010.01.0056

Definition 2.5. *[2] Let $f : \mathbb{T} \to \mathbb{R}$, $t \in \mathbb{T}^k$ and $\alpha \in (0,1]$. For $\alpha \in (0,1] \cap \{\frac{1}{q} | q$ is an odd number$\}$, we define $T_\alpha f(t) \in \mathbb{R}$ (provided it exists) with the property that for any $\epsilon > 0$, there exists $\delta > 0$ such that*

$$|[f(\sigma(t)) - f(s)] - T_\alpha f(t)\,[\sigma(t) - s]^\alpha| \leq \epsilon\,|\sigma(t) - s|^\alpha\,,$$

for all $s \in U_{\mathbb{T}}(t, \delta)$. The function $T_\alpha f(t)$ is called the conformable derivative (derivative, for short) of f of order α at t.

Remark 2.1. *It should be noted that by considering $\alpha = 1$ in Definition 2.5, we obtain the usual delta (or Hilger) derivative, as introduced in [3]. Furthermore, if $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} - \{m\}$, otherwise $\mathbb{T}^k = \mathbb{T}$.*

Definition 2.6. *[11] The function $f : \mathbb{T} \to \mathbb{R}_F$ is called regulated if its right-sided limits exist (finite) at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$.*

Definition 2.7. *[11] The function $f : \mathbb{T} \to \mathbb{R}_F$ is called rd-continuous provided it is continuous at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$.*

3 Fuzzy fractional derivative and integral on time scales

In this section, we present the concepts of fractional derivative and integral for fuzzy functions on time scales and conduct the study of their corresponding properties.

Definition 3.1. *Let $f : \mathbb{T} \to \mathbb{R}_F$ and $t \in \mathbb{T}^k$. For $\alpha \in (0,1] \cap \{\frac{1}{q} | q$ is an odd number$\}$, the fuzzy function f is called (α)-differentiable at t if for any $\epsilon > 0$, there exists a $\delta > 0$, and a neighborhood $U_{\mathbb{T}} = (t - \delta, t + \delta) \cap \mathbb{T}$ of t such that $t \mp h \in U_{\mathbb{T}}$ and $f(t + h) \ominus f(\sigma(t))$ and $f(\sigma(t)) \ominus f(t - h)$ exist and*

$$D\left(f(t+h) \ominus f(\sigma(t)), f^{(\alpha)}(t)(h - \mu(t))^\alpha\right) \leq \epsilon\,|h - \mu(t)|^\alpha\,,$$
$$D\left(f(\sigma(t)) \ominus f(t-h), f^{(\alpha)}(t)(h + \mu(t))^\alpha\right) \leq \epsilon(h + \mu(t))^\alpha,$$

with $0 \leq h < \delta$.

Theorem 3.1. *Let $t \in \mathbb{T}^k$ be right-dense and $\alpha \in (0,1] \cap \{\frac{1}{q} | q$ is an odd number$\}$. Then, the properties hold, as follows:*

i) If $f : \mathbb{T} \to \mathbb{R}_F$ is (α)-differentiable at t, then it is continuous at t.

(ii) The fuzzy function f is (α)-differentiable at t if and only if

$$f^{(\alpha)}(t) = \lim_{h \to 0^+} \frac{1}{h^\alpha}(f(t+h) \ominus f(t)) = \lim_{h \to 0^+} \frac{1}{h^\alpha}(f(t) \ominus f(t-h)),$$

provided that the corresponding limits exist for $h > 0$ small enough.

Theorem 3.2. *(1) Let $f, g : \mathbb{T} \to \mathbb{R}_F$ be (α)-differentiable at $t \in \mathbb{T}^k$, then $f + g : \mathbb{T} \to \mathbb{R}_F$ is (α)-differentiable at t and*

$$(f + g)^{(\alpha)}(t) = f^{(\alpha)}(t) + g^{(\alpha)}(t).$$

(2) Let $\lambda \in \mathbb{R}$ and f be (α)-differentiable at t, then $\lambda f : \mathbb{T} \to \mathbb{R}_F$ is (α)-differentiable at t and

$$(\lambda f)^{(\alpha)}(t) = \lambda f^{(\alpha)}(t).$$

Theorem 3.3. *Let $g : \mathbb{T} \to \mathbb{R}$ be (α)-differentiable at t and $c \in \mathbb{R}_F$. If $f : \mathbb{T} \to \mathbb{R}_F$ is defined by $f(t) = cg(t)$ such that $g(\sigma(t))T_\alpha g(t) > 0$, then f is (α)-differentiable at t and $f^{(\alpha)}(t) = cT_\alpha g(t)$.*

Example 3.1. *Let $\mathbb{T} = \mathbb{N}$ and $f : \mathbb{T} \to \mathbb{R}_F$ be defined by*

$$f(t) = (1,2,3)t^2,$$

for all $t \in \mathbb{T}$. Therefore, by Theorem 7 in [2] and Theorem 3.3, we have

$$\begin{aligned} f^{(\alpha)}(t) &= (1,2,3)(T_\alpha t^2) \\ &= (1,2,3)((t+1)^2 - t^2) \\ &= (1,2,3)(2t+1). \end{aligned}$$

We proceed with the definition of the (α)-derivative of f of order β, as follows:

Definition 3.2. *Let β be a non-negative real number. We define (α)-derivative of f of order β by*

$$f^{(\beta)} = (f^{\Delta^I})^{(\alpha)},$$

where $I := [\beta]$ (that is, I is the integer part of β) and $\alpha := \beta - I$.

Example 3.2. *Let $\mathbb{T} = 5\mathbb{N}$ and $\beta = 1.3$. If $f(t) = (1,2,3)t^2$, by Theorems 3.2,3.3, we obtain*

$$\begin{aligned} f^{(1.3)}(t) &= (f^\Delta)^{(0.3)} \\ &= \left((1,2,3)(t^2)^\Delta\right)^{(0.3)} \\ &= ((1,2,3)(2t+5))^{(0.3)} \\ &= (1,2,3)((2t)^{(0.3)} + 5^{(0.3)}) \\ &= 2(1,2,3)5^{0.7}. \end{aligned}$$

Now, we introduce the fractional integral for fuzzy functions on time scales.

Definition 3.3. *A regulated function $f : \mathbb{T} \to \mathbb{R}_F$ is said to be β-integral if, for $0 \le \beta \le 1$, we have*

$$\int f(t)\Delta^\beta t = \left(\int f(t)\Delta t\right)^{(1-\beta)} \in \mathbb{R}_F,$$

where $\int f(t)\Delta t$ is the Δ-integral, as introduced in [11].

Theorem 3.4. *Let $f, g : \mathbb{T} \to \mathbb{R}_F$ be rd-continuous. Then, we have*

$$\int (\lambda f(t) + g(t))\, \Delta^\beta t = \lambda \int f(t)\Delta^\beta t + \int g(t)\Delta^\beta t,$$

where $\lambda \in \mathbb{R}$ and $0 \le \beta \le 1$.

Example 3.3. *Let $f : \mathbb{N} \to \mathbb{R}_F$ be defined by $f(t) = (1,2,3)t$. Note that*

$$\int t\Delta t = \frac{t^2}{2} + C,$$

where C is constant. Therefore, for $t \in \mathbb{N}$, it follows that

$$\begin{aligned} \int (t,2t,3t)\Delta^\beta t &= \left(\int (t,2t,3t)\Delta t\right)^{(1-\beta)} \\ &= \left((\int t\Delta t)^{(1-\beta)}, (\int 2t\Delta t)^{(1-\beta)}, (\int 3t\Delta t)^{(1-\beta)} \right) \\ &= (0.5(2t+1), 2t+1, 1.5(2t+1)) \in \mathbb{R}_F. \end{aligned}$$

DOI: 10.5540/03.2023.010.01.0056

4 Conclusions

In this study, we have introduced the new definition of the fractional derivative and integral for fuzzy functions on time scales. Also, we have presented some properties of the introduced concepts. The results here might be used in further research for the study of fuzzy fractional differential equations on time scales.

Acknowledgements

The research has been partially supported by FAPESP under grants no. 2022/00196-1 and 2020/09838-0 and by CNPq under grant no. 313313/2020-2,314885/2021-8.

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