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A Constant Rank-type Constraint Qualification for Multi-Objective Continuous-Time Nonlinear Programming

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Abstract. The paper addresses multi-objective continuous-time nonlinear programming problems with equality and inequality constraints. It is obtained first and second-order necessary optimality conditions through a constant rank-type constraint qualification.

Keywords. Continuous-time programming, Efficient solutions, Second order conditions, Constraint qualifications, Constant rank condition.

1 Introdução

Multi-objective optimization problems constitute an important class of decision problems and have been often studied in recent decades. Such problems arise in several areas of science, such as engineering, economics, and administration, among others, and have as objective the decision-making involving multiple choice parameters. One of the precursors in the field of multi-objective optimization is Pareto who, in his famous work "Cours d'Economie Politique" [11], introduces the concept of an efficient solution. Conceptually, a point is called efficient (or Pareto) when it is not possible to improve any objective without worsening some other.

Several authors have studied and applied the concept of Pareto optimality to obtain necessary and sufficient conditions for multi-objective problems in finite-dimensional spaces (see [1, 3]). In continuous-time programming problems, to cite but a few, we cite the works [8–10, 12]. For example, in [9], de Oliveira established optimality conditions of saddle-point-type and some classical duality results. No differentiability assumption was imposed. In [10], de Oliveira and Rojas-Medar considered smooth problems and derived sufficient optimality conditions and duality results under generalized invexity.

More recently, Jović [4] presented necessary and sufficient optimality conditions under generalized concavity (quasiconcavity and pseudoconcavity) for vector continuous-time programming using the Mangasarian-Fromovitz constraint qualification given by Monte and de Oliveira [7]. In [5], Jović and Marinković derived zero order optimality conditions under convexity assumptions.

In the articles cited above, the problems present only inequality constraints. In addition, the optimality conditions furnished are of the first order. In this work, we derive optimality conditions of first and second order for multi-objective continuous-time optimization problems, with equality and inequality constraints. The optimality conditions were developed for efficient solutions, also

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known as Pareto solutions. A relaxed version of the constant rank-type constraint qualification given in Monte and de Oliveira [6] for the mono-objective case is used here for establishing the KKT-type conditions.

The work is organized in the following way. Some preliminaries are given in Section 2, where some important assumptions and the definition of efficient solutions are stated. In Section 3, a review of the scalar problem is performed, and a small (but important) adjustment is made to the results of Monte and de Oliveira [6]. In Section 4, Karush-Kuhn-Tucher necessary optimality conditions for continuous-time optimality problems are derived. Moreover, we remark that such conditions are valid under linear independence constraint qualification. An example is presented for illustration.

2 Preliminaries

The paper deals with the multi-objective continuous-time nonlinear programming problem posed as

$$\begin{array}{ll} \max & \displaystyle\int_0^T \phi(z(t),t)\,dt = \left(\int_0^T \phi_1(z(t),t)\,dt,\ldots,\int_0^T \phi_q(z(t),t)\,dt\right) \\ \text{s.t.} & h(z(t),t)=0 \text{ a.e. } t\in[0,T], \\ & g(z(t),t)\geq 0 \text{ a.e. } t\in[0,T], \\ & z\in L^\infty([0,T];\mathbb{R}^n), \end{array} \qquad \text{(MCTP)}$$

where $\phi:\mathbb{R}^n\times[0,T]\to\mathbb{R}^q$, $h:\mathbb{R}^n\times[0,T]\to\mathbb{R}^p$, $g:\mathbb{R}^n\times[0,T]\to\mathbb{R}^m$ are given functions and $\phi_k(z,t)$ denotes the k-th component of $\phi(z,t)\in\mathbb{R}^q$.

All vectors are column vectors. Prime denotes transposition. All integrals are in the Lebesgue sense. Inequality signs between vectors should be read component-wise.

The feasible set of problem (MCTP) is denoted by

$$\Omega=\{z\in L^\infty([0,T];\mathbb{R}^n): h(z(t),t)=0,\ g(z(t),t)\geq 0 \text{ a.e. } t\in[0,T]\}.$$

Set the index sets as $K=\{1,\ldots,q\}$, $I=\{1,\ldots,p\}$ and $J=\{1,\ldots,m\}$. The index set of all binding constraints at $\bar{z}\in\Omega$ is defined by

$$I_a(t)=\{j\in J: g_j(\bar{z}(t),t)=0\} \text{ a.e. } t\in[0,T].$$

B denotes the open unit ball with center at the origin in $\mathbb{R}^n$.

Given $\varepsilon>0$ and a reference solution $\bar{z}\in\Omega$, consider the following hypotheses:

(H1) For all $k\in K$, $\phi_k(z,\cdot)$ is measurable for each z, $\phi_k(\cdot,t)$ is twice continuously differentiable on $\bar{z}(t)+\varepsilon\bar{B}$ a.e. $t\in[0,T]$; there exists $K_\phi>0$ such that

$$\|\nabla\phi_k(\bar{z}(t),t)\|+\|\nabla^2\phi_k(\bar{z}(t),t)\|\leq K_\phi \text{ a.e. } t\in[0,T];$$

(H2) $h(z,\cdot)$ and $g(z,\cdot)$ are measurable for each z; $h(\cdot,t)$ and $g(\cdot,t)$ are twice continuously differentiable on $\bar{z}(t)+\varepsilon\bar{B}$ a.e. $t\in[0,T]$; $g(\bar{z}(\cdot),\cdot)$ is essentially bounded in $[0,T]$;

(H3) There exists an increasing function $\theta:(0,\infty)\to(0,\infty)$, $\theta(s)\downarrow 0$ as $s\downarrow 0$, such that for all $\tilde{z},z\in\bar{z}(t)+\varepsilon\bar{B}$,

$$\|\nabla(h,g)(\tilde{z},t)-\nabla(h,g)(z,t)\|\leq\theta(\|\tilde{z}-z\|) \text{ a.e. } t\in[0,T];$$

There exists $K_0>0$ such that

$$\|\nabla(h,g)(\bar{z}(t),t)\|+\|\nabla^2(h,g)(\bar{z}(t),t)\|\leq K_0 \text{ a.e. } t\in[0,T].$$

The maximization in (MCTP) is in the sense of an efficient point.

Definition 2.1. *$\bar{z} \in \Omega$ is said to be a local efficient solution for (MCTP) if there is no other $z \in \Omega$ with $z(t) \in \bar{z}(t) + \varepsilon B$ a.e. $t \in [0,T]$ for some $\varepsilon > 0$, such that*

$$\int_0^T \phi_k(z(t),t)\,dt \geq \int_0^T \phi_k(\bar{z}(t),t)\,dt, \;\; k \in K,$$

with at least one strict inequality.

3 The Scalar Case Revisited

When $q = 1$, $\phi = \phi_1$ and we have the scalar continuous-time problem

$$\begin{array}{ll} \max & \Phi(z) = \int_0^T \phi(z(t),t)\,dt \\ \text{s.t.} & h(z(t),t) = 0 \text{ a.e. } t \in [0,T], \\ & g(z(t),t) \geq 0 \text{ a.e. } t \in [0,T], \\ & z \in L^\infty([0,T];\mathbb{R}^n). \end{array} \tag{SCTP}$$

Definition 3.1. *$\bar{z} \in \Omega$ is said to be a local optimal solution for (SCTP) if there exists $\varepsilon > 0$ such that*

$$\int_0^T \phi(z(t),t)\,dt \leq \int_0^T \phi(\bar{z}(t),t)\,dt$$

for all $z \in \Omega$ with $z(t) \in \bar{z}(t) + \varepsilon B$ a.e. $t \in [0,T]$.

We can guarantee the validity of KKT-type necessary optimality conditions for (SCTP) assuming the constant rank-type constraint qualification (CRCQ) defined below.

Definition 3.2. *The constant rank constraint qualification (CRCQ) for the problem (SCTP) is said to be satisfied at $\bar{z} \in \Omega$ if the following two requirements are fulfilled:*

(i) There exist integer numbers $r_t = r(t)$ a.e. $t \in [0,T]$ and a real number $\varepsilon > 0$ such that $\text{rank}[M(z,w,t)] = r_t$ on $(\bar{z}(t),\bar{w}(t)) + \varepsilon B$ a.e. $t \in [0,T]$, where

$$M(z,w,t) = \begin{bmatrix} \nabla h(z,t) & 0 \\ \nabla g(z,t) & \text{diag}\{-2w_j\}_{j=1}^m \end{bmatrix},$$

with $\bar{w}_j(t) = \sqrt{g_j(\bar{z}(t),t)}$ a.e. in $[0,T]$, $j \in J$;

(ii) There exist an index subset, say $\{i_1,\ldots,i_{r_t}\}$, and a constant $C > 0$ such that

$$\det\{\Upsilon(t)\Upsilon(t)'\} \geq C \;\text{ a.e. } t \in [0,T],$$

where $\Upsilon(t)$ is the matrix obtained after removing from $M(\bar{z}(t),\bar{w}(t),t)$ the rows of index $i \notin \{i_1,\ldots,i_{r_t}\}$.

In Monte and de Oliveira [6], KKT-type necessary optimality conditions were provided under a slightly different constant rank condition: the integer number r_t in definition above were assumed to be constant with respect to t. Recently, we realized that, if the integer number r is allowed to vary with the parameter $t \in [0,T]$, then the results are still valid. With such changes, the following result is proved similarly to [6].

Proposition 3.1. *Let $\bar{z}$ be a local optimal solution of (SCTP). Suppose that the assumptions (H1)-(H3) and (CRCQ) are satisfied. Then, there exist $u \in L^\infty([0,T];\mathbb{R}^p)$ and $v \in L^\infty([0,T];\mathbb{R}^m)$ such that*

$$\nabla\phi(\bar{z}(t),t) + \sum_{i=1}^{p} u_i(t)\nabla h_i(\bar{z}(t),t) + \sum_{j=1}^{m} v_j(t)\nabla g_j(\bar{z}(t),t) = 0, \tag{1}$$

$$v_j(t) \geq 0,\ \ j \in J, \tag{2}$$

$$v_j(t)g_j(\bar{z}(t),t) = 0,\ \ j \in J, \tag{3}$$

for almost every $t \in [0,T]$, and

$$\int_0^T \gamma(t)'\Big[\nabla^2\phi(\bar{z}(t),t) \ + \ \sum_{i=1}^{p} u_i(t)\nabla^2 h_i(\bar{z}(t),t) + \sum_{j=1}^{m} v_j(t)\nabla^2 g_j(\bar{z}(t),t)\Big]\gamma(t)\,dt \leq 0, \tag{4}$$

for all $\gamma \in \mathcal{N}$, where

$$\mathcal{N} = \{\gamma \in L^\infty([0,T];\mathbb{R}^n) \ : \ \nabla h_i(\bar{z}(t),t)'\gamma(t) = 0,\ i \in I,\ a.e.\ t \in [0,T], \ \nabla g_j(\bar{z}(t),t)'\gamma(t) = 0,\ j \in I_a(t),\ a.e.\ t \in [0,T]\}.$$

4 Karush-Kuhn-Tucker Necessary Optimality Condition

In this section, we discuss the necessary optimality conditions for (MCTP). Let $\kappa \in K$ and $\bar{z} \in \Omega$. Consider the auxiliary problem below:

$$\begin{array}{ll} \max & \Phi_\kappa(z) = \displaystyle\int_0^T \phi_\kappa(z(t),t)\,dt \\ \text{s.t.} & \phi_k(z(t),t) \geq \phi_k(\bar{z}(t),t) \text{ a.e. } t \in [0,T],\ k \in K \setminus \{\kappa\}, \\ & z \in \Omega. \end{array} \tag{$\mathrm{P}_\kappa(\bar{z})$}$$

The following lemma shows the connection between (MCTP) and scalar problem $\mathrm{P}_\kappa(\bar{z})$, and plays a key role in proving the main result in this section.

Lemma 4.1 (Chankong and Haimes [2])**.** *If $\bar{z} \in \Omega$ is a local efficient solution for (MCTP), then $\bar{z}$ solves $(P_\kappa(\bar{z}))$ locally for all $\kappa \in K$.*

Now, we give necessary Karush-Kuhn-Tucker optimality conditions for (MCTP).

Theorem 4.1. *Let $\bar{z}$ be a local efficient solution for (MCTP) and suppose that assumptions (H1)-(H3) are valid. In addition, assume that (CRCQ) for $(P_{\bar{\kappa}}(\bar{z}))$ is satisfied at $\bar{z}$ for some $\bar{\kappa} \in K$. Then, there exist $\lambda \in L^\infty([0,T];\mathbb{R}^k)$, $u \in L^\infty([0,T];\mathbb{R}^p)$ and $v \in L^\infty([0,T];\mathbb{R}^m)$ such that*

$$\sum_{k=1}^{q} \lambda_k(t)\nabla\phi_k(\bar{z}(t),t) + \sum_{i=1}^{p} u_i(t)\nabla h_i(\bar{z}(t),t) + \sum_{j=1}^{m} v_j(t)\nabla g_j(\bar{z}(t),t) = 0, \tag{5}$$

$$v_j(t)g_j(\bar{z}(t),t) = 0,\ \ v_j(t) \geq 0,\ \ j \in J, \tag{6}$$

$$\sum_{k=1}^{q} \lambda_k(t) = 1,\ \ \lambda_k(t) \geq 0,\ \ k \in K, \tag{7}$$

for almost every $t \in [0,T]$*, and*

$$\int_0^T \gamma(t)'\Big[\sum_{k=1}^{q} \lambda_k(t)\nabla^2\phi_k(\bar{z}(t),t) + \sum_{i=1}^{p} u_i(t)\nabla^2 h_i(\bar{z}(t),t) + \sum_{j=1}^{m} v_j(t)\nabla^2 g_j(\bar{z}(t),t)\Big]\gamma(t)\,dt \leq 0 \tag{8}$$

for all $\gamma \in \mathcal{N}$*, where*

$$\begin{aligned}\mathcal{N} = \{\gamma \in L^\infty([0,T];\mathbb{R}^n) \ :\ & \nabla h_i(\bar{z}(t),t)'\gamma(t) = 0,\ i \in I,\ \textit{a.e. } t \in [0,T],\\ & \nabla g_j(\bar{z}(t),t)'\gamma(t) = 0,\ j \in I_a(t),\ \textit{a.e. } t \in [0,T],\\ & \nabla \phi_k(\bar{z}(t),t)'\gamma(t) = 0,\ k \in K\setminus\{\bar{\kappa}\},\ \textit{a.e. } t \in [0,T]\}.\end{aligned}$$

Proof. Since $\bar{z}$ is an efficient solution of (MCTP), by Lemma 4.1, $\bar{z}$ solves $(\mathrm{P}_\kappa(\bar{z}))$ for all $\kappa \in K$. By assumption, (CRCQ) for $(\mathrm{P}_{\bar{\kappa}}(\bar{z}))$ is satisfied at $\bar{z}$ for some $\bar{\kappa}$. Therefore, by Proposition 3.1, there exist $\tilde{\lambda} \in L^\infty([0,T];\mathbb{R}^{q-1})$, $\tilde{u} \in L^\infty([0,T];\mathbb{R}^p)$ and $\tilde{v} \in L^\infty([0,T];\mathbb{R}^m)$ such that, for almost every $t \in [0,T]$,

$$\nabla\phi_{\bar{\kappa}}(\bar{z}(t),t) + \sum_{k\in K\setminus\{\bar{\kappa}\}} \tilde{\lambda}_k(t)\nabla\phi_k(\bar{z}(t),t) + \sum_{i=1}^{p} \tilde{u}_i(t)\nabla h_i(\bar{z}(t),t) + \sum_{j=1}^{m} \tilde{v}_j(t)\nabla g_j(\bar{z}(t),t) = 0, \tag{9}$$

$$\tilde{v}_j(t) g_j(\bar{z}(t),t) = 0,\ j \in J, \tag{10}$$

$$\tilde{v}_j(t) \geq 0,\ j \in J, \tag{11}$$

$$\tilde{\lambda}_k(t) \geq 0,\ k \in K\setminus\{\bar{\kappa}\}, \tag{12}$$

and

$$\int_0^T \gamma(t)'\Big[\nabla^2\phi_{\bar{\kappa}}(\bar{z}(t),t) + \sum_{k\in K\setminus\{\bar{\kappa}\}} \tilde{\lambda}_k(t)\nabla^2\phi_k(\bar{z}(t),t) + \sum_{i=1}^{p} \tilde{u}_i(t)\nabla^2 h_i(\bar{z}(t),t) + \sum_{j=1}^{m} \tilde{v}_j(t)\nabla^2 g_j(\bar{z}(t),t)\Big]\gamma(t)\,dt \leq 0 \tag{13}$$

for all $\gamma \in \mathcal{N}$, where

$$\begin{aligned}\mathcal{N} = \{\gamma \in L^\infty([0,T];\mathbb{R}^n) \ :\ & \nabla h_i(\bar{z}(t),t)'\gamma(t) = 0,\ i \in I,\ \text{a.e. } t \in [0,T],\\ & \nabla g_j(\bar{z}(t),t)'\gamma(t) = 0,\ j \in I_a(t),\ \text{a.e. } t \in [0,T],\\ & \nabla \phi_k(\bar{z}(t),t)'\gamma(t) = 0,\ k \in K\setminus\{\bar{\kappa}\},\ \text{a.e. } t \in [0,T]\}.\end{aligned}$$

Let us define

$$\lambda_{\bar{\kappa}}(t) = \frac{1}{1 + \sum_{k\in K\setminus\{\bar{\kappa}\}} \tilde{\lambda}_k(t)} \ \text{a.e. } t \in [0,T].$$

Multiplying (9)-(11) and (13) by $\lambda_{\bar{\kappa}}(t)$ and setting $\lambda_k(t) = \lambda_{\bar{\kappa}}(t)\tilde{\lambda}_k(t)$, $k \in K\setminus\{\bar{\kappa}\}$, $v_j(t) = \lambda_{\bar{\kappa}}(t)\tilde{v}_j(t)$, $j \in J$, and $u_i(t) = \lambda_{\bar{\kappa}}(t)\tilde{u}_i(t)$, $i \in I$, for almost every $t \in [0,T]$, we conclude that conditions (5)-(8) hold. ☐

In the following, we have an illustrative example in which the assumptions of Theorem 4.1 are satisfied.

Example 4.1. *Consider the multi-objective problem*

$$\begin{array}{ll} \max & \Phi(z) = \left(\int_0^1 [-z_1(t)z_2(t)]\,dt, \int_0^1 [-z_1^2(t) - 2z_2(t)]\,dt \right) \\ s.t. & z_1(t) + z_2(t) - t = 0 \ \ a.e. \ t \in [0,1], \\ & -z_1(t) - z_2(t) + t = 0 \ \ a.e. \ t \in [0,1], \\ & z \in L^\infty([0,1];\mathbb{R}^2), \end{array}$$

along with the local efficient solution $\bar{z}(t) = \left(2 - \frac{t}{2}, -2 + \frac{3}{2}t\right)$ *a.e.* $t \in [0,1]$. *We will verify that (CRCQ) is satisfied at* $\bar{z}$ *for both* $(\mathrm{P}_1(\bar{z}))$ *and* $(\mathrm{P}_2(\bar{z}))$.

The problem $(\mathrm{P}_1(\bar{z}))$ *is given by*

$$\begin{array}{ll} \max & \Phi_1(z) = \int_0^1 [-z_1(t)z_2(t)]\,dt \\ s.t. & z_1(t) + z_2(t) = t \ \ a.e. \ t \in [0,1], \\ & -z_1(t) - z_2(t) + t = 0 \ \ a.e. \ t \in [0,1], \\ & -z_1^2(t) - 2z_2(t) \geq -\left(2 - \frac{t}{2}\right)^2 - 2\left(-2 + \frac{3}{2}t\right) \ \ a.e. \ t \in [0,1], \\ & z \in L^\infty([0,1];\mathbb{R}^2). \end{array}$$

We have that

$$M(z,w,t) = \begin{bmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ -2z_1 & -2 & -2w \end{bmatrix} \ \ a.e. \ t \in [0,1].$$

We see, for almost every $t \in [0,1]$, *that*

$$\mathrm{rank}(M(z,w,t)) = \begin{cases} 1 \ \textit{for} \ (z_1,z_2,w) = (1,z_2,0), \ \ z_2 \ \textit{free}, \\ 2 \ \textit{otherwise}. \end{cases}$$

Clearly, $\bar{w}(t) = 0$ *a.e. in* $[0,1]$. *By choosing* $\varepsilon > 0$ *small enough,*

$$\mathrm{rank}(M(z,w,t)) = 2 \ \forall (z,w) \in (\bar{z}(t), \bar{w}(t)) + \varepsilon B \ \ a.e. \ t \in [0,1].$$

Note that

$$M(\bar{z}(t), \bar{w}(t), t) = \begin{bmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ -4+t & -2 & 0 \end{bmatrix} \ \ a.e. \ t \in [0,1].$$

Let

$$\Upsilon(t) = \begin{bmatrix} 1 & 1 & 0 \\ -4+t & -2 & 0 \end{bmatrix} \ \ a.e. \ t \in [0,1].$$

Then, $\det(\Upsilon(t)\Upsilon(t)') = t^2 - 4t + 4 \geq 1$ *a.e. in* $[0,1]$. *Therefore, (CRCQ) for* $(\mathrm{P}_1(\bar{z}))$ *is satisfied at* $\bar{z}$. *Analysing the problem* $(\mathrm{P}_2(\bar{z}))$

$$\begin{array}{ll} \max & \Phi_1(z) = \int_0^1 [-z_1^2(t) - 2z_2(t)]\,dt \\ s.t. & z_1(t) + z_2(t) = t \ \ a.e. \ t \in [0,1], \\ & -z_1(t) - z_2(t) + t = 0 \ \ a.e. \ t \in [0,1], \\ & -z_1(t)z_2(t) \geq -\left(2 - \frac{t}{2}\right)\left(-2 + \frac{3}{2}t\right) \ \ a.e. \ t \in [0,1], \\ & z \in L^\infty([0,1];\mathbb{R}^2), \end{array}$$

analogously, we conclude that (CRCQ) for $(\mathrm{P}_2(\bar{z}))$ *is satisfied at* $\bar{z}$*. It is clear that the multi-objective problem satisfies assumptions (H1)-(H3). By Theorem 4.1, there exist* $\lambda \in L^\infty([0,1];\mathbb{R}^2)$ *and* $u \in L^\infty([0,1];\mathbb{R}^2)$ *such that (5)-(8) hold.*

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