

A simple algorithm for reliability evaluation of stochastic-flow networks under distance limitation

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Abstract. Various real-world systems, including computer and communication networks, transportation, and distribution systems, can be modeled as a stochastic/multistate flow network (MFN). Reliability indices are of great importance in evaluating the quality of service in MFNs. One such index is the two-terminal reliability (2TR), which represents the probability that the maximum flow in the network is not less than a certain demand level. Researchers have investigated the 2TR problem for MFNs, considering constraints such as budget, time, or distance limitations. Nevertheless, distance constraints have received relatively less attention in the literature, despite their significant role in optimizing the efficiency of some transmission networks. Hence, this study focuses on the 2TR problem with a distance constraint in an MFN and proposes a simple yet efficient algorithm to solve it. We demonstrate the effectiveness of our algorithm through a benchmark example.

Keywords. Multistate flow networks, System reliability, Distance limitation, Minimal paths,

1 Introduction

A stochastic/multistate flow network (MFN) is a type of network where the components and the entire network can exist in more than two states. In traditional network flow problems, the components, such as arcs or nodes, have only two states, typically representing a binary decision, such as whether the component is open or closed [5, 34]. In an MFN, however, each component can exist in multiple states, and the flow through each component depends on its state [15]. For example, in a transportation network, an arc may be in one of several states, depending on whether the road is closed for construction, has reduced capacity due to an accident, or is open with standard capacity[4]. Similarly, in a communication network, a transmission line may have multiple states, depending on whether it is usually working, has a partial failure, or has completely failed [29, 35].

The MFNs are useful in modeling real-world systems with varying reliability and performance, allowing for a more accurate representation of the system's behavior and more effective optimization of its performance. Computer and communication networks [24, 28, 29], transportation systems [4, 23, 30], distribution systems [3, 25], and rework networks [20] are some instances of such networks.

Out of many reliability indices for MFNs, the two-terminal reliability (2TR) index at a given demand level of d , denoted by R_d , is a critical factor in designing and optimizing the performance of many real-world systems. Engineers and designers can make informed decisions on network design, maintenance, and operation by accurately assessing the R_d to ensure reliable and efficient performance. In an MFN, the index R_d represents the probability of successfully transmitting at least d units of flow (such as data, commodities, or goods) from a source node to a sink node. Various direct and indirect algorithms have been proposed in the literature, ranging from accurate methods to approximation algorithms to calculate R_d [2, 6–8, 10, 11, 14, 16–19, 22, 33].

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Researchers often include additional constraints, such as time and cost, to make the 2TR problem more practical for real-world applications. Besides time and cost, transmission distance limitation is a significant factor that impacts the performance of an MFN. The distance between nodes in the network can affect the flow of goods or commodities and, consequently, the overall efficiency and reliability of the network [5, 36]. For example, consider a transportation network where a delivery truck needs to travel long distances to reach its destination. The longer the distance, the more chances of delay or breakdown, leading to a higher risk of failure and lower performance of the transportation network. Therefore, optimizing the transmission distance is critical in designing and managing MFNs for various real-world applications, such as communication, transportation, and distribution networks.

It is worth noting that the distance constraint has been predominantly investigated in the reliability evaluation of binary-state flow networks (BFN) [5] and has received less attention in the context of multistate flow networks (MFN) [36]. The authors in [5] suggested a new algorithm for detecting and deleting irrelevant edges in the reliability assessment of a BFN under distance constraint. The authors then provided experiments on different real-world topologies to demonstrate that the algorithm could produce substantial computational gains. The authors in [36] have presented a strategy to remove the irrelevant arcs in an MFN and proposed an approximation algorithm to evaluate the 2TR, which can be considered an approach for the 2TR problem under distance constraints. The reliability of an MFN at a demand level of d under distance limitation of λ , denoted by $R_{(d,\lambda)}$, is the probability of transmitting at least d units of flow from the source node to the sink node through only the MPs with their length not greater than λ .

The common point about the studies on BFN and MFN is that scholars have usually defined irrelevant arcs as arcs with no contribution to flow transmission under distance restrictions, then presented some strategies to detect and remove those arcs from the network. And finally, they computed the reliability of the new network after removing the irrelevant arcs. However, besides some technical issues with the definitions related to the irrelevant arcs and the provided strategies for their detection, it is worth noting that the computational cost of those strategies is often expensive. Hence, in this work, we propose a different strategy that does not need to detect or remove any irrelevant arc and directly sets the flow on the minimal paths (MPs) whose length is greater than the given distance limit. In the next section, we provide some preliminaries, introduce a new concept (d, λ) -MP and present our simple yet efficient approach to address the problem and illustrate it through a benchmark example.

2 Main block

Consider a multistate flow network (MFN) denoted by $G(N, A, M, L)$, where $N = 1, 2, \dots, n$ represents the set of nodes and $A = a_1, a_2, \dots, a_m$ represents the set of arcs in the network. The vector $M = (M_1, M_2, \dots, M_m)$ contains the maximum capacity of each arc a_i , with M_i denoting the maximum capacity of arc a_i , for $i = 1, 2, \dots, m$. The vector $L = (l_1, l_2, \dots, l_m)$ contains the length of each arc a_i , with l_i denoting the length of arc a_i , for $i = 1, 2, \dots, m$. Here, n and m represent the number of nodes and arcs in the network, respectively. Nodes 1 and n are designated as the source and sink nodes. For instance, in Fig. 1, $N = \{1, 2, 3, 4, 5\}$ is the set of nodes, $A = \{a_1, \dots, a_8\}$ is the set of arcs, and $M_1 = 3$ and $l_1 = 1$ show respectively the maximum capacity and the length of arc a_1 . Let x_i be the current capacity of arc a_i with values in $\{0, 1, \dots, M_i\}$, forming the current system state vector (SSV) $X = (x_1, \dots, x_m)$. For instance, $X = (2, 2, 0, 1, 0, 0, 1, 0, 0)$ can be an SSV in Fig. 1. A path is a sequence of adjacent arcs connecting nodes 1 and n in the network. A minimal path (MP) is a path whose proper subsets are not paths. For instance, $P = \{a_1, a_4, a_3, a_5\}$ is a path but not an MP in Fig. 1 while $P_1 = \{a_1, a_5\}$ is an MP. Let p be the number of all the MPs in the network. For a minimal path

of P_j , $LP_j = \sum_{i: a_i \in P_j} l_i$ is its length, and $CP_j(X) = \min\{x_i | a_i \in P_j\}$ is its capacity under SSV X . For instance, $LP_1 = l_1 + l_5 = 1 + 2 = 3$ and $CP_1(M) = \min\{M_1, M_5\} = 2$ are respectively the length and the capacity of $P_1 = \{a_1, a_5\}$ in Fig. 1. Let $e_i = (0, \dots, 0, 1, 0, \dots, 0)$ be an SSV in which the capacity level is 1 for a_i and 0 for the other arcs. Let also $V(X)$ be the maximum flow of the network from node 1 to node n under X , d be a non-negative integer number less than or equal to $V(M)$ giving the required flow to be sent from node 1 to node n in the network, and λ be the maximum acceptable distance for flow transmission from node 1 to n .

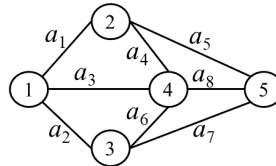


Figure 1: A benchmark network example with $M = (3, 2, 2, 1, 2, 1, 3, 2)$ and $L = (1, 2, 1, 3, 2, 1, 2, 1)$ taken from [13].

Our work is based on the following assumptions: (1) Each node is deterministic and perfectly reliable. (2) The flow in the network satisfies the flow conservation law [1]. (3) The capacity of each arc $a_i \in A$ is a random integer value between 0 and M_i . (4) The capacities of the arcs are statistically independent. (5) Every arc belongs to at least one MP from node 1 to node n .

The relationship between two system state vectors, X and Y , is denoted as $X \leq Y$ if every component of X is less than or equal to the corresponding component of Y . If $X \leq Y$ and there exists at least one component for which X is strictly less than Y , then $X < Y$. A vector $X \in \Psi$ is minimal if there is no vector $Y \in \Psi$ such that $Y < X$.

Without considering the distance limitation, the concepts of d -MP and d -MP candidate have been defined as follows in the literature [9, 13, 21, 26, 27, 31]. Letting $F = (f_1, \dots, f_p)$ be a solution of

$$\begin{cases} (i) f_1 + f_2 + \dots + f_p = d, \\ (ii) 0 \leq f_j \leq \min\{K_j, d\}, \quad j = 1, 2, \dots, p, \\ (iii) \sum_{j: a_i \in P_j} f_j \leq M_i, \quad i = 1, 2, \dots, m, \end{cases} \quad (1)$$

a system state vector $X = (x_1, x_2, \dots, x_m)$, obtained using the following equation, is called the associated d -MP candidate with the flow vector of F .

$$x_i = \sum_{j: a_i \in P_j} f_j, \quad \forall i = 1, 2, \dots, m. \quad (2)$$

In simpler terms, given a solution to the system (1), one can obtain a state vector through (2), and this vector is called the d -MP candidate. Moreover, a state vector $X = (x_1, x_2, \dots, x_m)$ is a d -MP if and only if $V(X) = d$ and $V(X - e_i) < d$, for each i with $x_i > 0$ [9]. One can show that every minimal vector in the set of all the d -MP candidates is a d -MP [21]. However, a more efficient approach to check a candidate for being a d -MP, given below as Lemma 2.1, was presented in [32] and improved in [9].

Lemma 2.1. *A d -MP candidate is a d -MP if and only if there is no directed cycle in the network under it.*

We now extend these definitions and results to the case with distance limitation λ .

Definition 2.1. A d -MP (candidate) is a (d, λ) -MP (candidate) if and only if under it at least d units of flow can be transmitted from node 1 to node n through only the MPs with their length less than or equal to λ .

It is time-consuming if one finds all the d -MPs and then checks each for the distance limitation. To tackle it, Zhang and Shao [36] introduced the concept of irrelevant arcs for the MFNs for two cases; (1) the arcs that do not belong to any MP and (2) the arcs that only belong to MPs with length greater than λ , which is the distance limitation. Then, the authors proposed an approach to detect and remove the irrelevant arcs from the network. However, one must verify all the MPs to check every arc for being irrelevant, which is also time-consuming. Here, we define the irrelevant MP (IMP) concept to address the problem more efficiently.

Definition 2.2. The MP P_j is irrelevant if its length is greater than the distance limitation. That is P_j is irrelevant when $LP_j = \sum_{i: a_i \in P_j} l_i > L$.

Now, one can compute the MPs' lengths once to determine the IMPs and set zero the flow on all the IMPs in calculating all the d -MPs. This way, the obtained d -MPs are indeed the (d, λ) -MPs according to Definition 2.1 because there is no flow on any IMP. Thus, the following lemma is at hand.

Lemma 2.2. Any (d, λ) -MP is a d -MP under which the flow on IMPs is zero and vice-versa.

We now propose our algorithm to find all the (d, λ) -MPs in an MFN.

Algorithm 1

Input: $G(N, A, M, L)$, its MPs, the demand level of d , and the distance limitation of λ .

Output: All the (d, λ) -MPs.

Step 1. Calculate $LP_j = \sum_{i|a_i \in P_j} l_i$, for $j = 1, 2, \dots, p$, and let $I = \{j | LP_j > L\}$ and $J = \{j | LP_j \leq L\}$. Compute $KP_j(M)$ for $j \in J$. Let $S = \phi$.

Step 2. Find a solution F by solving the following system:

$$\begin{cases} (i) \sum_{j \in J} f_j = d, \\ (ii) 0 \leq f_j \leq \min\{KP_j(M), d\}, \quad j \in J. \end{cases} \quad (3)$$

If no more solutions are found, stop the algorithm and output the set of obtained (d, λ) -MPs (S). Otherwise, if $I \neq \phi$, then set $f_j = 0$, for $j \in I$.

Step 3. Use Eq. (2) to calculate the SSV X corresponding to F . If $X \not\leq M$, it is not a (d, λ) -MP, then go back to Step 2 to find the next solution.

Step 4. If there is a directed cycle in $G(N, A, X, L)$, then go to Step 2 to find the next solution.

Step 5. If X is not duplicated, it is a (d, λ) -MP, then add it to S .

One notes that: (1) The difference between the system (3) and the first two items in the system (1) is that the system (3) is limited on the MPs with the indices belonging to J . (2) The item (iii) in the system (1) is not used in the system (3) because it is more efficient first to find all solutions of a simple constrained Diophantine equation, which is solving the system (3), and then calculate the associated state vector with each solution to check it for inequality (iii), which is $X \leq M$. Steps 2 and 3 in our proposed algorithm do it. (3) Step 4 in the algorithm is based on Lemma 2.1. (4) There is a probability of finding duplicate (d, λ) -MPs from different flow vectors [12], and hence Step 5 in the algorithm checks every solution not to be repetitive. Therefore, the following theorem is at hand.

Theorem 2.1. Algorithm 1 determines all the (d, λ) -MPs without any duplicates.

One effective way to better comprehend a new approach is by observing its application in a straightforward benchmark network example.

Example 2.1. We use Algorithm 1 to find all the $(6, 6)$ -MP in the network given in Fig. 1 with $M = (3, 2, 2, 1, 2, 1, 3, 2)$ and $L = (1, 2, 1, 3, 2, 1, 2, 1)$.

Solution: The demand level is $d = 6$ and the distance limit is $\lambda = 6$. There are nine MPs in the network: $P_1 = \{a_1, a_5\}$, $P_2 = \{a_2, a_7\}$, $P_3 = \{a_3, a_8\}$, $P_4 = \{a_1, a_4, a_8\}$, $P_5 = \{a_2, a_6, a_2\}$, $P_6 = \{a_3, a_4, a_5\}$, $P_7 = \{a_3, a_6, a_7\}$, $P_8 = \{a_1, a_4, a_6, a_7\}$, and $P_9 = \{a_2, a_6, a_4, a_5\}$.

Step 1. We calculate the MPs' lengths as follows. $LP_1 = 3$, $LP_2 = 4$, $LP_3 = 2$, $LP_4 = 5$, $LP_5 = 4$, $LP_6 = 6$, $LP_7 = 4$, $LP_8 = 7$, and $LP_9 = 8$. As $\lambda = 6$, then we have $I = \{8, 9\}$ and $J = \{1, 2, \dots, 7\}$. Now, calculating the capacities of the MPs with their indices belonging to J , the capacities of the first seven MPs are respectively 5, 5, 4, 6, 5, 5, and 6.

Step 2. The system (3) has a total of 84 possible solution vectors denoted as $F = (f_1, \dots, f_7)$. If we set $f_8 = f_9 = 0$, we get 84 solution vectors that satisfy the system (1). However, only the corresponding SSVs to six of these 84 vectors are $(6, 6)$ -MP candidates. Hence, we only present these six solutions for clarity: $(2, 2, 0, 1, 0, 0, 1, 0, 0)$, $(1, 2, 1, 1, 0, 0, 1, 0, 0)$, $(2, 2, 1, 0, 0, 0, 1, 0, 0)$, $(2, 1, 1, 1, 0, 0, 1, 0, 0)$, $(2, 2, 2, 0, 0, 0, 0, 0, 0)$, and $(2, 2, 1, 1, 0, 0, 0, 0, 0)$.

Step 3. Here are the SSVs associated with the six solutions obtained in Step 2, and all of them satisfy the condition $X \leq M$: $(3, 2, 1, 1, 2, 1, 3, 1)$, $(2, 2, 2, 1, 1, 1, 3, 2)$, $(2, 2, 2, 0, 2, 1, 3, 1)$, $(3, 1, 2, 1, 2, 1, 2, 2)$, $(2, 2, 2, 0, 2, 0, 2, 2)$, and $(3, 2, 1, 1, 2, 0, 2, 2)$.

Step 4. The network has a directed cycle for none of the obtained candidates. Thus, all of them are $(6, 6)$ -MP.

Step 5. There is no repetitive solution; hence, the solution set contains all the calculated candidates in Step 3.

3 Conclusion

In this study, we investigated the two-terminal reliability problem with a distance constraint in a multistate flow network and proposed a simple yet efficient algorithm to solve the problem. We showed the correctness of the algorithm and illustrated it through a benchmark example with five nodes and eight arcs.

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