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Non singularity criteria for non strictly diagonally dominant pentadiagonal matrices

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Abstract. Square matrices, A, strictly diagonally dominant belong to an important class of invertible matrices that have an LU decomposition. We will present in this work new non singularity criteria based on Crout's method for non strictly diagonally dominant pentadiagonal (or tridiagonal) matrices that admit an LU decomposition. These criteria are simple and easy to implement. There are many papers on this subject in the literature. However, the results that ensure non singularity of A usually depend on the conditions that are not promptly obtained.

Palavras-chave. Crout's method, pentadiagonal matrices, non strictly diagonally dominant matrices

1 Introduction

Linear systems with tridiagonal and pentadiagonal matrices can arise from differential equation discretizations referring to mathematical models in different areas of science. The coefficient matrix of such a linear system must be non singular to ensure unicity of solution. Next, we mention some papers that study non singular tridiagonal or pentadiagonal matrix.

The development of algorithms for finding the inverse of any general non singular tridiagonal or pentadiagonal matrix, [6], [7], [5], and [2] (see also the references in these articles) is a subject that has been studied by many authors. The results of these research usually depend on the existence of the LU factorization of a non sigular matrix A, such that $A = LU$. Besides, most of those articles assume that the matrix is invertible or present non trivial conditions that ensure the non singularity of A and its LU factorization.

In our work, we are going to consider non singularity criteria for non strictly diagonally dominant pentadiagonal matrices. Consequently, we are going to obtain extremely simple sufficient conditions for existence of the LU factorization of those matrices. These conditions are extremely simple because they do not require the computationally expensive calculations of determinants, for example, but they only require equalities and inequalities between the matrix coefficients (see Definition 2.3).

We cite three works concerning these issues: [1, 3, 4]. The two first papers are recent, from 2023, and the third is from 2003. In the first paper [1], the authors have present a sufficient condition for existence of the LU factorization of a Toeplitz symmetric tridiagonal matrix A. They used an analysis based on the parameters of Crout's method, and concluded that $\det(A) \neq 0$. In the second paper, the author has characterized, in terms of combinatorial structure and sign pattern, when a weakly (non strictly) diagonally dominant matrix may be invertible. In the third paper [4], the author has presented necessary and sufficient conditions for non singularity of the non strictly block diagonally dominant matrices.

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In our work, we have developed a low-cost test for detecting in a simple way when a weakly diagonally dominant pentadiagonal matrix is non singular and has an LU decomposition. Our sufficient conditions, specifically for pentadiagonal matrix, are simpler than the ones presented in the aforementioned works [3, 4].

This test based on Crout's method uses criteria that are presented in Definition 2.3. The following result is important: Theorems 3.1.

2 Definitions and Preliminary Results

The following Lemmas will be needed throughout the paper for the proof of important results associated with pentadiagonal matrices.

Lemma 2.1. *Let d, γ_k, and b_k, $1 \le k \le n$, be real numbers. Then,*

$$\left| d - \sum_{k=1}^{n} \gamma_k b_k \right| \ge |d| - \sum_{k=1}^{n} |\gamma_k||b_k|.$$

Proof. The proof follows by mathematical induction and observing that:

$$|d| = |d - \gamma b + \gamma b| \le |d - \gamma b| + |\gamma b| \quad \longrightarrow \quad |d - \gamma b| \ge |d| - |\gamma||b|,$$

for any real number d, γ and b. □

Lemma 2.2. *Let a, b, c, d, and e real numbers that satisfy: $|d| \ge |a| + |b| + |c| + |e|$ and $d \ne 0$. Suppose the real numbers α, β, γ, ϵ, $\tilde{\gamma}$, $\tilde{\epsilon}$, $\overline{\epsilon}$, $\overline{\gamma}$ are such that: (i) $\alpha = d - \tilde{\gamma}\beta - \overline{\epsilon}\, e \ne 0$, (ii) $\beta = b - \overline{\gamma}\, e$, (iii) $\gamma = \frac{a - \tilde{\epsilon}\,\beta}{\alpha}$, (iv) $|\gamma| + |\epsilon| \le 1$, (v) $|\tilde{\gamma}| + |\tilde{\epsilon}| \le 1$, and (vi) $|\overline{\gamma}| + |\overline{\epsilon}| \le 1$. If $|\gamma| = 1$, $a \ne 0$, $b = c = 0$, then $sgn(a\, d) = sgn(\gamma)$, where $sgn(x) = x/|x|$, if $x \ne 0$, and $sgn(0) = 0$.*

Proof. Since $b = c = 0$, it follows that $|d| \ge |a| + |e|$, $\gamma = \frac{a + \tilde{\epsilon}\,\overline{\gamma}\, e}{\alpha}$, and $\alpha = d + (\tilde{\gamma}\,\overline{\gamma} - \overline{\epsilon})\, e$.

If $|\gamma| = 1$, then $a + \tilde{\epsilon}\,\overline{\gamma}\, e = \alpha$ or $a + \tilde{\epsilon}\,\overline{\gamma}\, e = -\alpha$. Thus, considering $\lambda_1 = \overline{\epsilon} + \overline{\gamma}(\tilde{\epsilon} - \tilde{\gamma})$ and $\lambda_2 = \overline{\epsilon} - \overline{\gamma}(\tilde{\epsilon} + \tilde{\gamma})$, it follows that $d = a + e\,\lambda_1$, or $-d = a - e\,\lambda_2$. According to the lemma's hypotheses, we obtain that $|\lambda_1| \le 1$ and $|\lambda_2| \le 1$; hence, $|d| \le |a| + |e|$. Therefore, $|d| = |a| + |e|$.

Now, suppose that $\gamma = 1$. Thus,

$$a^2 + 2\,a\,e\,\lambda_1 + \lambda_1^2\, e^2 = d^2 = a^2 \, + \, 2|a|\,|e| \, + \, e^2 \longrightarrow$$

$$2\,a\,e\,\lambda_1 = 2|a|\,|e| \, + \, (1 - \lambda_1^2)\, e^2 \ge 0.$$

Hence, $a\,d = a^2 + a\,e\lambda_1 > 0$ and, therefore, $sgn(a\,d) = sgn(\gamma)$.

If $\gamma = -1$, then

$$a^2 - 2\,a\,e\,\lambda_2 \, + \, \lambda_2^2\, e^2 = d^2 = a^2 \, + \, 2\,|a|\,|e| \, + \, e^2 \, \longrightarrow$$

$$-2\,a\,e\,\lambda_2 = 2\,|a|\,|e| + (1 \, - \, \lambda_2^2)\, e^2 \ge 0 \, \longrightarrow \, a\,e\,\lambda_2 \le 0.$$

Hence, $a\,d = -a^2 \, + \, a\,e\lambda_2 < 0$ and, therefore, $sgn(a\,d) = sgn(\gamma)$. □

Lemma 2.3. *Let a, b, c, d, and e be real numbers that satisfy: $|d| \ge |a| + |b| + |c| + |e|$, and $d \ne 0$. Suppose the real numbers β, γ, ϵ, $\tilde{\gamma}$, $\tilde{\epsilon}$, α are such that: (i) $\beta = b - \tilde{\gamma} e$, (ii) $|\gamma| + |\epsilon| \le 1$, (iii) $|\tilde{\gamma}| + |\tilde{\epsilon}| \le 1$, and (iv) $\alpha = d - \gamma\beta - \tilde{\epsilon} e \ne 0$. Then*

(I) $\dfrac{|a - \epsilon\beta| + |c|}{|\alpha|} \le 1.$

(II) $|d| > |a| + |b| + |c| + |e| \longrightarrow \dfrac{|a - \epsilon\beta| + |c|}{|\alpha|} < 1.$

(III) $e^2 + b^2 \neq 0,\ |\gamma| + |\epsilon| < 1,\ |\tilde{\gamma}| + |\tilde{\epsilon}| < 1 \longrightarrow \dfrac{|a - \epsilon\beta| + |c|}{|\alpha|} < 1.$

(IV) $e \neq 0,\ |\tilde{\gamma}| + |\tilde{\epsilon}| < 1 \longrightarrow \dfrac{|a - \epsilon\beta| + |c|}{|\alpha|} < 1.$

Proof. (I) Since $|\gamma|+|\epsilon| \leq 1$, it follows that $|\epsilon| \leq (1-|\gamma|)$. Therefore, $|a-\epsilon\beta|+|c| \leq |a|+|\epsilon||\beta|+|c| \leq |a| + (1 - |\gamma|)|\beta| + |c|$. According to Lemma 2.1 and remembering that $|d| \geq |a| + |b| + |c| + |e|$, we obtain that $|a| + (1 - |\gamma|)|\beta| + |c| \leq |d| - |b| - |e| + |\beta| - |\gamma||\beta| \leq |d| + |\beta - b| - |e| - |\gamma||\beta|$. Note that $(|\tilde{\gamma}| - 1) \leq -|\tilde{\epsilon}|$ and $\beta - b = -\tilde{\gamma}e$. Therefore, using once again Lemma 2.1, we obtain that $|d| + |\beta - b| - |e| - |\gamma||\beta| \leq |d| + |\tilde{\gamma}||e| - |e| - |\gamma||\beta| = |d| + (|\tilde{\gamma}| - 1)|e| - |\gamma||\beta| \leq |d| - |\tilde{\epsilon}||e| - |\gamma||\beta| \leq |d - \gamma\beta - \tilde{\epsilon}e| = |\alpha|$, from where the result follows.

(II) Use the demonstration from the previous item (I), although considering the following strict inequality: $|a| + (1 - |\gamma|)|\beta| + |c| < |d| - |b| - |e| + |\beta| - |\gamma||\beta|$.

(III) Suppose that $e \neq 0$. Hence, similar to the demonstration of item (I), $|a - \epsilon\beta| + |c| \leq |a| + |\epsilon||\beta| + |c| \leq |a| + (1 - |\gamma|)|\beta| + |c| \leq |d| - |b| - |e| + |\beta| - |\gamma||\beta| \leq |d| + |\beta - b| - |e| - |\gamma||\beta|$. Note that $(|\tilde{\gamma}| - 1) < -|\tilde{\epsilon}|$ and $\beta - b = -\tilde{\gamma}e$. Thus, we obtain that $|d| + |\beta - b| - |e| - |\gamma||\beta| \leq |d|+|\tilde{\gamma}||e|-|e|-|\gamma||\beta| = |d|+(|\tilde{\gamma}|-1)|e|-|\gamma||\beta| < |d|-|\tilde{\epsilon}||e|-|\gamma||\beta| \leq |d-\gamma\beta-\tilde{\epsilon}e| = |\alpha|$, from where the result follows. Additionally, suppose that $e = 0$, $b \neq 0$; hence, $\beta = b$ and $\alpha = d-\gamma b$. Therefore, $|a - \epsilon\beta| + |c| = |a - \epsilon b| + |c| \leq |a| + |\epsilon||b| + |c| < |a| + (1 - |\gamma|)|b| + |c| \leq |d| - |\gamma||b| \leq |d - \gamma b| = |\alpha|$, from where the result follows.

(IV) The same demonstration as the first part of the previous item. $\square$

The notation $M_{n\times n}(\mathbb{R})$ represents the set of all matrices of order n with elements in $\mathbb{R}$. In this work, we consider pentadiagonal matrices, $A \in M_{n\times n}(\mathbb{R})$, according to the following definition.

Definition 2.1. *A matrix $A = (A_{ij})$ of order n is pentadiagonal if $A_{ij} = 0$ whenever $|i - j| > 2$.*

If $A \in \mathbf{P} = \{A \in M_{n\times n}(\mathbb{R}); A \text{ is pentadiagonal}\}$, then this matrix is represented by:

$$A = \begin{pmatrix} d_1 & a_1 & c_1 & 0 & 0 & 0 & \dots & & & & 0 \\ b_2 & d_2 & a_2 & c_2 & 0 & 0 & & & & & \vdots \\ e_3 & b_3 & d_3 & a_3 & c_3 & 0 & & & & & \\ 0 & e_4 & b_4 & d_4 & a_4 & c_4 & & & & & \\ & & \ddots & \ddots & \ddots & & & & & & \\ & & & & & & \ddots & \ddots & & & \\ \vdots & & & & & & & & & & \\ 0 & & & & & & e_{n-2} & b_{n-2} & d_{n-2} & a_{n-2} & c_{n-2} \\ & & & & & & & e_{n-1} & b_{n-1} & d_{n-1} & a_{n-1} \\ 0 & \dots & & & & & & & e_n & b_n & d_n \end{pmatrix}. \tag{1}$$

In this case, we consider $b_1 = e_1 = e_2 = c_{n-1} = c_n = a_n = 0$.

The diagonally dominant matrices are also important for this work, and are defined as follows.

Definition 2.2. *A matrix $A = (A_{ij})$ of order n is diagonally dominant if, and only if, for all i, $1 \leq i \leq n$,*

$$|A_{ii}| \geq \sum_{j=1, j\neq i}^{n} |A_{ij}|.$$

In Definition 2.2, we can replace the symbol $\geq$ by the symbol $>$. In this case, we say that A is a strictly diagonally dominant matrix. It is known that every strictly diagonally dominant matrix is non singular and has LU decomposition.

Next, the set P_D is composed by pentadiagonal matrices which are diagonally dominant. We will show that these matrices are non singular and have LU decomposition.

Definition 2.3. *The set P_D is defined as the set of pentadiagonal matrices A (see Equation (1)) such that their diagonal elements satisfy: $d_i \neq 0$, $|d_i| \geq |e_i|+|b_i|+|a_i|+|c_i|, i \in \{1,\dots,n\}$. Besides, the elements on each row of A must satisfy one of the following conditions:*

- *(a) $b_i = e_i = 0$; or*
- *(b) $|d_i| > |e_i| + |b_i| + |a_i| + |c_i|$; or*
- *(c) $b_i^2 + e_i^2 \neq 0$, $|d_i| = |e_i| + |b_i| + |a_i| + |c_i|$, and $a_i^2 + c_i^2 \neq 0$; or*
- *(d) $b_i \neq 0$, $a_i = c_i = 0$, $|d_i| = |e_i|+|b_i|+|a_i|+|c_i|$, and $|d_{i-1}| > |e_{i-1}|+|b_{i-1}|+|a_{i-1}|+|c_{i-1}|$; or*
- *(e) $e_i \neq 0$, $a_i = c_i = 0$, $|d_i| = |e_i|+|b_i|+|a_i|+|c_i|$, and $|d_{i-2}| > |e_{i-2}|+|b_{i-2}|+|a_{i-2}|+|c_{i-2}|$; or*
- *(f) $e_i = 0$, $b_i \neq 0$, $b_{i-1} = 0$, $a_{i-1} \neq 0$, $c_{i-1} = 0$ and $sgn(b_i.d_i) = -sgn(a_{i-1}.d_{i-1})$.*

Remark: Every strictly diagonally dominant pentadiagonal matrix belongs to the set P_D, according to the item (b) from Definition 2.3.

Let A be a squared matrix of order n which has an LU decomposition, where $L_{ii} \neq 0$ and $U_{ii} = 1$, $1 \leq i \leq n$. It is well known that A^T also has an $\mathcal{LU}$ decomposition, where $\mathcal{L}_{ii} \neq 0$ and $\mathcal{U}_{ii} = 1$, $1 \leq i \leq n$.

Remark: Considering the previous result, the subjects of our studies are pentadiagonal matrices A, such that A or A^T belongs to the set P_D.

Example 1. The following matrix A is not diagonally dominant (observe the first and third rows of A). However, its transpose A^T is diagonally dominant. Besides, note that rows 1 to 5 from the transpose matrix satisfy the conditions presented in set P_D: row 1 - item (a); rows $2, 3$ and 4 - item (c); row 5 - item b. Therefore, by Theorem 3.1, A^T belongs to set P_D and, consequently, A has an LU decomposition, and $\det(A) \neq 0$.

$$A = \begin{bmatrix} 2 & 3 & 1 & 0 & 0 \\ 1 & 5 & 0 & 0 & 0 \\ 1 & 1 & 3 & 1 & 1 \\ 0 & -1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix} \text{ and } A^T = \begin{bmatrix} 2 & 1 & 1 & 0 & 0 \\ 3 & 5 & 1 & -1 & 0 \\ 1 & 0 & 3 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 0 & 2 \end{bmatrix}.$$

3 Main Results

Let A be a pentadiagonal matrix as shown in Equation (1). According to [7], if $A = LU$, then L and U are pentadiagonal matrices given by

$$L = \begin{bmatrix} \alpha_1 & 0 & & & \dots & 0 \\ \beta_2 & \alpha_2 & 0 & & & \vdots \\ e_3 & \beta_3 & \alpha_3 & 0 & & \\ & & \ddots & \ddots & \ddots & \\ \vdots & & & & & 0 \\ 0 & \dots & & e_n & \beta_n & \alpha_n \end{bmatrix} \quad and\ U = \begin{bmatrix} 1 & \gamma_1 & \epsilon_1 & 0 & \dots & 0 \\ 0 & 1 & \gamma_2 & \epsilon_2 & 0 & \vdots \\ & 0 & 1 & \gamma_3 & \epsilon_3 & \\ & & & \ddots & \ddots & \\ \vdots & & & & 1 & \gamma_{n-1} \\ 0 & \dots & & & 0 & 1 \end{bmatrix}, \tag{2}$$

where

$$\alpha_i = \begin{cases} d_1, & i = 1; \\ d_2 - \gamma_1\beta_2, & i = 2; \\ d_i - \gamma_{i-1}\beta_i - \epsilon_{i-2}e_i, & i \in \{3, 4, \dots, n\}; \end{cases} \tag{3}$$

$$\gamma_i = \begin{cases} \dfrac{a_1}{\alpha_1}, & i = 1; \\ \\ \dfrac{a_i - \epsilon_{i-1}\beta_i}{\alpha_i}, & i \in \{2, 3, \dots, n-1\}; \end{cases} \tag{4}$$

$$\epsilon_i = \left\{ \dfrac{c_i}{\alpha_i}, \quad i \in \{1, 2, \dots, n-2\}; \right. \tag{5}$$

$$\beta_i = \begin{cases} b_2, & i = 2; \\ b_i - \gamma_{i-2}e_i, & i \in \{3, 4, \dots, n\}. \end{cases} \tag{6}$$

Crout's decomposition (see [7]) is possible whenever $\alpha_i \neq 0$, $1 \leq i \leq n$ and, in this case, $\det(A) \neq 0$.

In next Theorem 3.1, we will show that the matrices belonging to P_D are non singular.

Theorem 3.1. *If $A \in P_D$ (see Definition 2.3), then $A = LU$ and* $\det(A) \neq 0$.

Proof. It will be shown that $\alpha_i \neq 0$, $1 \leq i \leq n$. Consult the Equations (3), (4), (5), and (6) for $\alpha, \gamma, \epsilon, \beta$ definitions.

Note that $\alpha_1 = d_1 \neq 0$; hence $|\alpha_1| = |d_1| \geq |a_1| + |c_1|$ and $|\gamma_1| + |\epsilon_1| = |\frac{a_1}{\alpha_1}| + |\frac{c_1}{\alpha_1}| \leq 1$. If $|d_1| > |a_1| + |c_1|$, then $|\gamma_1| + |\epsilon_1| < 1$.

If $b_2 = e_2 = 0$ and $|d_2| \geq |a_2| + |c_2|$, then $|\alpha_2| = |d_2| > 0$. Thus, $\alpha_2 \neq 0$ and, according to Lemma 2.3, item (I), $|\gamma_2| + |\epsilon_2| \leq 1$. Considering item (II) from this same lemma, $|\gamma_2| + |\epsilon_2| < 1$ if $|d_2| > |a_2| + |c_2|$.

Suppose that $b_2 \neq 0$ and $|d_2| > |b_2| + |a_2| + |c_2|$. Knowing that $|\gamma_1| \leq 1$ and considering Lemma 2.1, we obtain that $|\alpha_2| = |d_2 - b_2\gamma_1| \geq |d_2| - |\gamma_1||b_2| \geq |d_2| - |b_2| > |a_2| + |c_2| \geq 0$. In this way, $|\alpha_2| > 0$ and, according to Lemma 2.3, item (II), $|\gamma_2| + |\epsilon_2| = |\frac{a_2 - \epsilon_1\beta_2}{\alpha_2}| + |\frac{c_2}{\alpha_2}| < 1$.

If $b_2 \neq 0$ and $|d_2| = |b_2| + |a_2| + |c_2|$ and $a_2^2 + c_2^2 \neq 0$, then, considering the same arguments presented previously, $|\alpha_2| = |d_2 - b_2\gamma_1| \geq |d_2| - |\gamma_1||b_2| \geq |d_2| - |b_2| = |a_2| + |c_2| > 0$. Thus, $|\alpha_2| > 0$ and, according to Lemma 2.3, item (I), $|\gamma_2| + |\epsilon_2| = |\frac{a_2 - \epsilon_1\beta_2}{\alpha_2}| + |\frac{c_2}{\alpha_2}| \leq 1$.

If $b_2 \neq 0$, $a_2 = c_2 = 0$, $|d_2| = |b_2| + |a_2| + |c_2|$, and $|d_1| > |a_1| + |c_1|$, then, according to Lemma 2.3, item (II), $|\gamma_1| + |\epsilon_1| = |\frac{a_1}{\alpha_1}| + |\frac{c_1}{\alpha_1}| < 1$. Knowing that $b_2 \neq 0$ and considering Lemma 2.1, we obtain that $|\alpha_2| = |d_2 - b_2\gamma_1| \geq |d_2| - |\gamma_1||b_2| > |d_2| - |b_2| = |a_2| + |c_2| = 0$. Therefore, $|\alpha_2| > 0$ and, by Lemma 2.3, item (I), $|\gamma_2| + |\epsilon_2| = |\frac{a_2 - \epsilon_1\beta_2}{\alpha_2}| + |\frac{c_2}{\alpha_2}| \leq 1$.

In order to prove by induction, suppose that $|\gamma_i| + |\epsilon_i| \leq 1$ and $\alpha_i \neq 0, \forall i, 1 \leq i \leq m$, with $m \geq 2$. It is important to note that $\alpha_{m+1} = d_{m+1} - \beta_{m+1}\gamma_m - \epsilon_{m-1}e_{m+1}$.

If $b_{m+1} = e_{m+1} = 0$ and $|d_{m+1}| \geq |a_{m+1}| + |c_{m+1}|$, then $|\alpha_{m+1}| = |d_{m+1}| > 0$. Thus, $\alpha_{m+1} \neq 0$ and, according to Lemma 2.3, item (I), $|\gamma_{m+1}| + |\epsilon_{m+1}| \leq 1$. Considering item (II) from this same lemma, $|\gamma_{m+1}| + |\epsilon_{m+1}| < 1$ if $|d_{m+1}| > |a_{m+1}| + |c_{m+1}|$.

Suppose $b_{m+1}^2 + e_{m+1}^2 \neq 0$ and $|d_{m+1}| > |e_{m+1}| + |b_{m+1}| + |a_{m+1}| + |c_{m+1}|$. If $|\gamma_m||\gamma_{m-1}| \leq |\gamma_{m-1}|$, then $|\gamma_m||\gamma_{m-1}| + |\epsilon_{m-1}| \leq |\gamma_{m-1}| + |\epsilon_{m-1}| \leq 1$. Thus, according to Lemma 2.1, $|\alpha_{m+1}| = |d_{m+1} - \beta_{m+1}\gamma_m - \epsilon_{m-1}e_{m+1}| \geq |d_{m+1}| - |\gamma_m||\beta_{m+1}| - |\epsilon_{m-1}||e_{m+1}| \geq |d_{m+1}| - |\gamma_m|(|b_{m+1}| + |\gamma_{m-1}||e_{m+1}|) - |\epsilon_{m-1}||e_{m+1}| \geq |d_{m+1}| - |b_{m+1}| - |e_{m+1}| > |a_{m+1}| + |c_{m+1}| \geq 0$. Hence, $|\alpha_{m+1}| > 0$ and, according to Lemma 2.3, item (II), $|\gamma_{m+1}| + |\epsilon_{m+1}| = |\frac{a_{m+1} - \epsilon_m\beta_{m+1}}{\alpha_{m+1}}| + |\frac{c_{m+1}}{\alpha_{m+1}}| < 1$.

If $b_{m+1}^2 + e_{m+1}^2 \neq 0$, $|d_{m+1}| = |e_{m+1}| + |b_{m+1}| + |a_{m+1}| + |c_{m+1}|$, and $a_{m+1}^2 + c_{m+1}^2 \neq 0$, then, considering the same arguments presented previously, we obtain that $|\alpha_{m+1}| = |d_{m+1} - \beta_{m+1}\gamma_m - \epsilon_{m-1}e_{m+1}| \geq |d_{m+1}| - |\gamma_m||\beta_{m+1}| - |\epsilon_{m-1}||e_{m+1}| \geq |d_{m+1}| - |\gamma_m|(|b_{m+1}| + |\gamma_{m-1}||e_{m+1}|) - |\epsilon_{m-1}||e_{m+1}| \geq |d_{m+1}| - |b_{m+1}| - |e_{m+1}| = |a_{m+1}| + |c_{m+1}| > 0$. Thus, $|\alpha_{m+1}| > 0$ and, according to Lemma 2.3, item (I), $|\gamma_{m+1}| + |\epsilon_{m+1}| = |\frac{a_{m+1} - \epsilon_m\beta_{m+1}}{\alpha_{m+1}}| + |\frac{c_{m+1}}{\alpha_{m+1}}| \leq 1$.

If $b_{m+1} \neq 0$, $a_{m+1} = c_{m+1} = 0$, $|d_{m+1}| = |e_{m+1}| + |b_{m+1}| + |a_{m+1}| + |c_{m+1}|$, and $|d_m| > |e_m| + |b_m| + |a_m| + |c_m|$, then, using the induction hypothesis and Lemma 2.3, item (II), we obtain that $|\gamma_m| + |\epsilon_m| = |\frac{a_m - \epsilon_{m-1}\beta_m}{\alpha_m}| + |\frac{c_m}{\alpha_m}| < 1$. In this way, $|\gamma_m||\gamma_{m-1}| \leq |\gamma_{m-1}|$. Hence, $|\gamma_m||\gamma_{m-1}| + |\epsilon_{m-1}| \leq |\gamma_{m-1}| + |\epsilon_{m-1}| \leq 1$. Knowing that $b_{m+1} \neq 0$ and considering Lemma 2.1, it is evident that $|\alpha_{m+1}| = |d_{m+1} - \beta_{m+1}\gamma_m - \epsilon_{m-1}e_{m+1}| \geq |d_{m+1}| - |\gamma_m||\beta_{m+1}| - |\epsilon_{m-1}||e_{m+1}| \geq |d_{m+1}| - |\gamma_m|(|b_{m+1}| + |\gamma_{m-1}||e_{m+1}|) - |\epsilon_{m-1}||e_{m+1}| > |d_{m+1}| - |b_{m+1}| - |e_{m+1}| = |a_{m+1}| + |c_{m+1}| = 0$. Thus, $|\alpha_{m+1}| > 0$ and, according to Lemma 2.3, item (I), $|\gamma_{m+1}| + |\epsilon_{m+1}| = |\frac{a_{m+1} - \epsilon_m\beta_{m+1}}{\alpha_{m+1}}| + |\frac{c_{m+1}}{\alpha_{m+1}}| \leq 1$.

If $e_{m+1} \neq 0$, $a_{m+1} = c_{m+1} = 0$, $|d_{m+1}| = |e_{m+1}| + |b_{m+1}| + |a_{m+1}| + |c_{m+1}|$, and $|d_{m-1}| > |e_{m-1}| + |b_{m-1}| + |a_{m-1}| + |c_{m-1}|$, then, using the induction hypothesis and Lemma 2.3, item (II), we obtain that $|\gamma_{m-1}| + |\epsilon_{m-1}| < 1$ (keeping in mind that $b_1 = e_1 = e_2 = 0$ and $\beta_2 = b_2$). Thus, $|\gamma_m||\gamma_{m-1}| \leq |\gamma_{m-1}|$, and $|\gamma_m||\gamma_{m-1}| + |\epsilon_{m-1}| \leq |\gamma_{m-1}| + |\epsilon_{m-1}| < 1$. Hence, $|\alpha_{m+1}| = |d_{m+1} - \beta_{m+1}\gamma_m - \epsilon_{m-1}e_{m+1}| \geq |d_{m+1}| - |\gamma_m||\beta_{m+1}| - |\epsilon_{m-1}||e_{m+1}| \geq |d_{m+1}| - |\gamma_m||b_{m+1}| - (|\gamma_m||\gamma_{m-1}| + |\epsilon_{m-1}|)|e_{m+1}| > |d_{m+1}| - |b_{m+1}| - |e_{m+1}| = |a_{m+1}| + |c_{m+1}| = 0$. Therefore, $|\alpha_{m+1}| > 0$ and, according to Lemma 2.3, item (IV), $|\gamma_{m+1}| + |\epsilon_{m+1}| = |\frac{a_{m+1} - \epsilon_m\beta_{m+1}}{\alpha_{m+1}}| + |\frac{c_{m+1}}{\alpha_{m+1}}| < 1$.

Suppose that $e_{m+1} = 0$, $b_{m+1} \neq 0$, $b_m = 0$, $a_m \neq 0$, $c_m = 0$, and $sgn(b_{m+1}.d_{m+1}) = -sgn(a_m.d_m)$, where $m > 2$, then $\alpha_{m+1} = d_{m+1} - \beta_{m+1}\gamma_m - \epsilon_{m-1}e_{m+1} = d_{m+1} - \gamma_m\, b_{m+1}$. Thus, if $|\gamma_m| < 1$, then $|\alpha_{m+1}| \geq |d_{m+1}| - |\gamma_m||b_{m+1}| > |d_{m+1}| - |b_{m+1}| \geq 0$. Hence, $\alpha_{m+1} \neq 0$. If $|\gamma_m| = 1$, then, according to Lemma 2.2, $-sgn(b_{m+1}.d_{m+1}) = sgn(a_m.d_m) = sgn(\gamma_m)$. In this way, if $\gamma_m = 1$, then b_{m+1} and d_{m+1} will have opposite signs. However, if $\gamma_m = -1$, then b_{m+1} and d_{m+1} will have the same sign. Therefore, in both cases, $\alpha_{m+1} \neq 0$ and, according to Lemma 2.3, item (I), $|\gamma_{m+1}| \leq 1$.

Therefore, by mathematical induction, it is possible to conclude that $\alpha_i \neq 0$, $1 \leq i \leq n$. $\square$

4 Conclusion

In our work, we have developed a low-cost test for detecting in a simple way when a weakly (non strictly) diagonally dominant pentadiagonal matrix is non singular and has an LU decomposition. This test based on Crout's method used criteria that are presented in Definition 2.3. In a future work, we will prove that if the reverse-permuted of the matrix A belongs to the set P_D, then the matrix A will have Crout's decomposition and $\det(A) \neq 0$ (see the **Example 2** below).

Example 2 (Pentadiagonal reverse-permuted matrix).

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 2 \end{bmatrix} \text{ and } \mathcal{A} = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}.$$

A does not belong to the set P_D because the third row of A does not satisfy none of the six items $((a) - (f))$ described in Definition 2.3. However, rows 1 to 6 from the reverse-permuted matrix $\mathcal{A}$ satisfy the conditions presented in set P_D: rows $1, 2$, and 4 - item (a); rows 3 and 5 - item (c); row 6 - item f. Therefore, by Theorem 3.1, the matrix $\mathcal{A}$ belongs to the set P_D.

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References

[1] C. G. Almeida and S. A. E. Remigio. "Sufficient Conditions for Existence of the LU Factorization of Toeplitz Symmetric Tridiagonal Matrices". In: **Trends in Computational and Applied Mathematics** (2023). Accepted. DOI: 10.5540/tcam.2022.024.01.00177.

[2] S. S. Askar and A. A. Karawia. "On Solving Pentadiagonal Linear Systems via Transformations". In: **Mathematical Problems in Engineering** 2015 (2015). Ed. by George S. Dulikravich. DOI: 10.1155/2015/232456.

[3] C. R. Johnson, C. Marijuán, and M. Pisonero. "Diagonal dominance and invertibility of matrices". In: **Special Matrices** 11.1 (2023). DOI: 10.1515/spma-2022-0181.

[4] L. Y. Kolotilina. "Nonsingularity/singularity criteria for nonstrictly block diagonally dominant matrices". In: **Linear Algebra and its Applications** 359 (2003), pp. 133–159. DOI: 10.1016/S0024-3795(02)00422-6.

[5] J. M. McNally. "A fast algorithm for solving diagonally dominant symmetric pentadiagonal Toeplitz systems". In: **Journal of Computational and Applied Mathematics** 234.4 (2010), pp. 995–1005. ISSN: 0377-0427. DOI: 10.1016/j.cam.2009.03.001.

[6] M. E. A. El-Mikkawy. "On the inverse of a general tridiagonal matrix". In: **Applied Mathematics and Computation** 150.3 (2004), pp. 669–679. ISSN: 0096-3003. DOI: 10.1016/s0096-3003(03)00298-4.

[7] X. L. Zhao and T. Z. Huang. "On the inverse of a general pentadiagonal matrix". In: **Applied Mathematics and Computation** 202.2 (2008), pp. 639–646. ISSN: 0096-3003. DOI: 10.1016/j.amc.2008.03.004.

