Proceeding Series of the Brazilian Society of Applied and Computational Mathematics, Vol. 3, N. 1, 2015.
Trabalho apresentado no XXXV CNMAC, Natal-RN, 2014.

Linking Fractional Derivative and Derivative in Time Scales

Berenice C. Damasceno **Luciano Barbanti**
Depto. de Matemática, FEIS, UNESP
15385-000, Ilha Solteira, SP
E-mails: berenice@mat.feis.unesp.br barbanti@mat.feis.unesp.br

Abstract: *Despite the increasingly number of works considering fractional derivatives and derivatives on time scales some basic facts are not yet taking place in the literature. We will be dealing with two of these facts, one of them concerning the minimal time transference of points in a process, and the another showing that the fractional derivative of monomials is in fact an entire derivative considered on an appropriate time scale.*

The theory of derivatives of fractional order goes back to Leibniz in 1695 when, in a note sent to L'Hospital, he discussed the meaning of the derivative of order half (1/2).

That note represented in fact the emergence of the theory of integrals and derivatives of an arbitrary order, which for three centuries has been treated as a purely theoretical mathematical field [3]. In the late nineteenth century this theory takeoff due to the works by Liouville, Grünwald, Letnikov and Riemann and during the twentieth century, mainly from the 70's, this new theory blow-up, as long as it was applied in experimental sciences.

The main applications of fractional differential equations are related to the model considering, for instance, the diffusion processes [1] in heterogeneous and anisotropic media [5], the electrochemical processes [7] the viscoelastic fluids [9], the electrical circuits [8], the biological systems [6], among others.

The Fractional Differential Equations are an excellent instrument for the description of memory and hereditary properties of various materials and processes [2].

This is the main advantage of the derivatives of non-integer order compared with those obtained from whole order.

However such effects are neglected for the most part in the current literature.

In this work we are pointing two of those advantages. We will be showing a result concerning minimal time transference of points in a process. This result is an apparent paradox along with the classical controllability theory, and should be explained .The another is by showing that the fractional derivative of the monomial $f(t) = t^s$ is in fact the derivative $\frac{d}{dt}$ considered on an appropriate time scale. We can expand the result in a straightforward way to functions under the Taylor form.

Fractional order derivatives

There are several concepts of fractional derivative.The most considered fractional order derivatives in the literature are the Caputo and the Riemann-Liouville and the Grünwald-Letnikov ones.

Here we will consider the Caputo fractional order derivatives, because it allows to be defining integer order initial conditions for Fractional Order Differential Equations.

Definition 1: Let be α a nonnegative real number with $m - 1 < \alpha \leq m\,, m\epsilon\mathbb{N}$, and a mapping $x: \mathbb{R} \to \mathbb{R}$.

The Caputo derivative of x is defined, for $t \geq 0$, as:

$$D^{\alpha}x(t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-s)^{m-\alpha+1}x^{(m)}(s)ds$$

The following result appears immediately from the definition 1:

***Proposition 1*:** For every nonnegative real numbers α , β we have that:

- if c is a constant then $D^{\alpha}c = 0$,

and for every $s > 0$,

- $D^{\alpha}\,(t^{s}) = \frac{\Gamma(s+1)}{\Gamma(s-\alpha+1)}\mathrm{t}^{s-\alpha}$, and

- $D^{\alpha+\beta}(t^{s}) = D^{\alpha}(D^{\beta}\,(t^{s}))$.

A result on reachable sets

Consider the ODE of the first order

$\dot{x} = u(t)$,

where

$u\epsilon L_1^{loc}(\mathbb{R}),\ \ \|u(t)\| \leq 1$.

The *set of reachability from zero* in the system is the set

$$R(t) = \{x(t); t \geq 0; x(0) = 0\} = [-t, t] \subset \mathbb{R}$$

Let be $0< \alpha < 1$. Consider the following problem:

$$\begin{cases} D^{\alpha}(D^{1-\alpha}x) = 1 \\ \quad x(0) = 0 \end{cases}$$

or:

$$\begin{cases} D^{1-\alpha}x = y \\ D^{\alpha}(y) = 1 \\ x(0) = 0 \end{cases} \quad ,$$

and one can make :

$$y(0) = k\,\epsilon\mathbb{R}$$

Proposition 2: If $k\epsilon\mathbb{R}$ then the mapping $x(t) = t + \frac{k}{\Gamma(2-\alpha)}t^{1-\alpha}$ is a solution of the problem above.

Corollary.1: If $k > 0$ then for every $t > 0$, there exists a value $0 < t_0 < t$ such that $x(t_0) = t$. Thus, due to the continuity of y , we have $\{x(t); y(0) = 0\} \supsetneq R(t)$.

Symmetrically, considering the negative part, we have

DOI: 10.5540/03.2015.003.01.0042

Proposition 3: If $k \epsilon \mathbb{R}$ then the mapping $x(t) = -t + \frac{k}{\Gamma(2-\alpha)} t^{1-\alpha}$ is a solution of the controlled system and: if $k < 0$ then for every $t > 0$, there exists a value $0 < t_0 < t$ such that $y(t_0) = -t$.

Time scales

Settled in 1988 by Stefan Hilger [4], the Calculus on time scales was created to unify the theories of the continuous time and the discrete dynamical systems.
(Here we take the notations by Hilger).

A time scale $\mathbb{T}$ is a closed non-void subset of the real numbers $\mathbb{R}$. Examples of time scales are: $\mathbb{R}, \mathbb{Z}$, $\{\frac{1}{2^k}\,; k \in \mathbb{N}\}$, or the Cantor set.

The fundamental operators are, for $t \in \mathbb{T}$:

$\sigma(t) = \inf\{r; r > t\}$ *-forward advance*

$\rho(t) = \sup\{r; r < t\}$ *–backward advance*

The Δ *−derivative is*

$$f^{\Delta}(t) = \begin{cases} f'(t) & if\ t\ is\ dense\ at\ right \\ \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t} & if\ t\ is\ isolated\ at\ right \end{cases}$$

Linking fractional derivative and entire derivative on a time scale $\mathbb{T}$

The main result in this section is:

Proposition 4: For a suitable $\mathbb{T}$, the equation

$$D^{\alpha}(t^s) = \frac{\Gamma(s+1)}{\Gamma(s-\alpha+1)} \mathrm{t}^{s-\alpha}$$

is in fact the equation :

$$\frac{d}{dt}(t^s) = \frac{\Gamma(s+1)}{\Gamma(s-\alpha+1)} \mathrm{t}^{s-\alpha} ,$$

for every $t \in \mathbb{T}$.

Proof: denote a point in $\mathbb{T}$ as r. We can construct the point $\sigma(r)$ by considering

$$\frac{\sigma(r)^s - r^s}{\sigma(r) - r} = \frac{\Gamma(s+1)}{\Gamma(s-\alpha+1)} r^{s-\alpha}. \tag{1}$$

The implicit function theorem allows us to consider a unique function $\sigma(r)$ for which (1) is true. By taking the fixed points r_0 in $\sigma(r) = r$, then every strictly increasing (or alternatively, strictly decreasing) sequence of numbers r converging to r_0 can be choose as being the time scale $\mathbb{T}$.

Example:

Take

$\alpha = \frac{1}{2}$, $s = 2$.

Then we construct the point $\sigma(r)$, for every $r \epsilon \mathbb{T}$, by considering

$$\frac{\sigma(r)^2 - r^2}{\sigma(r) - r} = \frac{3}{2} r^{3/2}$$

In this way we have the equation

$$\sigma(r) = \frac{3}{2} r^{3/2} - r \tag{2}$$

Following the previous notations we take $r_0 = 0$ and $\mathbb{T}$ as composed by a decreasing sequence of nonnegative point that satisfies (2). Observe that in this way the fractional differential ordinary equation

$$D^{1/2}\ (t^2) = \frac{\Gamma(2+1)}{\Gamma(2 - 1/2 + 1)} \mathrm{t}^{3/2}$$

is in fact a “normal” first order ODE

$$\frac{d}{dt}(t^2) = \frac{\Gamma(2+1)}{\Gamma(2-1/2+1)} \mathrm{t}^{3/2}$$

on the time scale $\mathbb{T}$.

References

[1] J. Crank, The Mathematics of Diffusion. Oxford University Press: New York, 1980.

[2] W. Deng, Short memory principle and a predictor-corrector approach for fractional differential equations. Journal of Computational and Applied Mathematics. Vol. 206, pp.174-188, (2007).

[3] R. Hilfer, Applications of Fractional Calculus in Physics. World Scientific: Singapore, 2000.

[4] S. Hilger, Analysis on measure chains - a unified approach to continuous and discrete calculus, Results Math.vol.18, pp.18-56, (1990).

[5] E. K. Lenzi, R. S. Mendes, G. Gonçalves, M. K. Lenzi and L. R. Silva, Fractional diffusion equation and green function approach: exact solutions. Physica A. vol. 360, pp.215-226 (2006).

[6] P. Macheras, Modeling in Biopharmaceutics and Pharmakinetics Dover Publications, New York, 2003.

[7] K. B. Oldham and J. Spanier, The Fractional Calculus. Dover Publications: New York, 2006.

[8] I. Plodubny, Fractional Differential Equations, Academic New York, 1999.

[9] P. J. Torvik and R. L. Bagley, On the appearance of the fractional derivative in the behavior of real materials, ASME J. Appl. Mech.,vol. 51, pp. 294–298. (1984).

DOI: 10.5540/03.2015.003.01.0042