Proceeding Series of the Brazilian Society of Applied and Computational Mathematics, Vol. 3, N. 1, 2015.
Trabalho apresentado no XXXV CNMAC, Natal-RN, 2014.

On Poincaré-Bendixson Theorem in Planar Nonsmooth Vector Fields

Tiago de Carvalho
Departamento de Matemática, Faculdade de Ciências, UNESP
17033-360, Bauru, SP
E-mail: tcarvalho@fc.unesp.br,

Claudio A. Buzzi, **Rodrigo D. Euzébio**,
Depto de Matemática, IBILCE, UNESP,
15054-000, São José do Rio Preto, SP
E-mail: buzzi@ibilce.unesp.br, rodrigo.euzebio@sjrp.unesp.br.

Abstract: *In this paper some qualitative and geometric aspects of nonsmooth vector fields theory are discussed. In the class of nonsmooth systems, that do not present sliding regions, a Poincaré-Bendixson Theorem is presented. The concepts of limit sets and minimal sets for nonsmooth systems are defined and compared with the classical ones.*

Keyword: *nonsmooth vector fields, Poincaré-Bendixson theory, minimal sets, limit sets.*

1 Introduction

Nonsmooth vector fields (NSVFs, for short) have become certainly one of the common frontiers between Mathematics and Physics or Engineering. Many authors have contributed to the study of NSVFs (see for instance the pioneering work [2] or the didactic works [1, 4], and references therein about details of these multi-valued vector fields). In our approach Filippov's convention is considered. So, the vector field of the model is discontinuous across a *switching manifold* and it is possible for its trajectories to be confined onto the switching manifold itself. The occurrence of such behavior, known as sliding motion, has been reported in a wide range of applications. We can find important examples in electrical circuits having switches, in mechanical devices in which components collide into each other, in problems with friction, sliding or squealing, among others.

For planar smooth vector fields there is a very developed theory nowadays. This theory is based in some important results. A now exhaustive list of such results include: *The Existence and Uniqueness Theorem*, *Hartman-Grobman Theorem*, *Poincaré-Bendixson Theorem* and *The Peixoto Theorem* among others. A very interesting and useful subject is to answer if these results are true or not at the NSVFs scenario. It is already known that the first theorem is not true (see Example 1 below) and the last theorem is true (under suitable conditions, see [3]).

The specific topic addressed in this paper concern with a Poincaré-Bendixson Theorem for NSVFs. In smooth vector fields, under relatively weak hypothesis, Poincaré-Bendixson Theorem tells us which kind of limit set can arise on an open region of the Euclidean space $\mathbb{R}^2$. In particular, minimal sets in smooth vector fields are contained in the limit sets (this fact can not be observed in NSVFs as we show below). As far as we know, in the context of NSVFs, this theme has not been treated in the literature until now.

2 Setting the problem

Let V be an arbitrarily small neighborhood of $0 \in \mathbb{R}^2$. We consider a codimension one manifold Σ of $\mathbb{R}^2$ given by $\Sigma = f^{-1}(0)$, where $f : V \to \mathbb{R}$ is a smooth function having $0 \in \mathbb{R}$ as a regular value (i.e. $\nabla f(p) \neq 0$, for any $p \in f^{-1}(0)$). We call Σ the *switching manifold* that is the separating boundary of the regions $\Sigma^+ = \{q \in V \mid f(q) \geq 0\}$ and $\Sigma^- = \{q \in V \mid f(q) \leq 0\}$. We can assume, locally around the origin of $\mathbb{R}^2$, that $f(x, y) = y$.

Designate by χ the space of C^r-vector fields on $V \subset \mathbb{R}^2$, with $r \geq 1$ large enough for our purposes. Call Ω the space of vector fields $Z : V \to \mathbb{R}^2$ such that

$$Z(x,y) = \begin{cases} X(x,y), & \text{for} \quad (x,y) \in \Sigma^+, \\ Y(x,y), & \text{for} \quad (x,y) \in \Sigma^-, \end{cases} \tag{1}$$

where $X = (X_1, X_2), Y = (Y_1, Y_2) \in \chi$. The trajectories of Z are solutions of $\dot{q} = Z(q)$ and we accept it to be multi-valued at points of Σ. The basic results of differential equations in this context were stated by Filippov in [2].

Consider Lie derivatives

$$X.f(p) = \langle \nabla f(p), X(p) \rangle \qquad \text{and} \qquad X^i.f(p) = \left\langle \nabla X^{i-1}.f(p), X(p) \right\rangle, i \geq 2$$

where $\langle .,. \rangle$ is the usual inner product in $\mathbb{R}^2$.

We distinguish the following regions on the discontinuity set Σ:

(i) $\Sigma^c \subseteq \Sigma$ is the *sewing region* if $(X.f)(Y.f) > 0$ on Σ^c .

(ii) $\Sigma^e \subseteq \Sigma$ is the *escaping region* if $(X.f) > 0$ and $(Y.f) < 0$ on Σ^e.

(iii) $\Sigma^s \subseteq \Sigma$ is the *sliding region* if $(X.f) < 0$ and $(Y.f) > 0$ on Σ^s.

The *sliding vector field* associated to $Z \in \Omega$ is the vector field Z^s tangent to Σ^s and defined at $q \in \Sigma^s$ by $Z^s(q) = m - q$ with m being the point of the segment joining $q + X(q)$ and $q + Y(q)$ such that $m - q$ is tangent to Σ^s (see Figure 1). It is clear that if $q \in \Sigma^s$ then $q \in \Sigma^e$ for $-Z$ and then we can define the *escaping vector field* on Σ^e associated to Z by $Z^e = -(-Z)^s$. In what follows we use the notation Z^Σ for both cases. In our pictures we represent the dynamics of Z^Σ by double arrows.

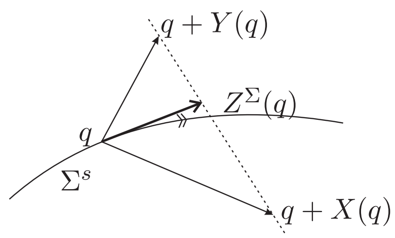

Figura 1: Filippov's convention.

We say that $q \in \Sigma$ is a Σ-*regular point* if $(X.f(q))(Y.f(q)) > 0$ or $(X.f(q))(Y.f(q)) < 0$ and $Z^\Sigma(q) \neq 0$ (i.e., $q \in \Sigma^e \cup \Sigma^s$ and it is not an equilibrium point of Z^Σ).

The points of Σ which are not Σ-regular are called Σ-*singular*. We distinguish two subsets in the set of Σ-singular points: Σ^t and Σ^p. Any $q \in \Sigma^p$ is called a *pseudo-equilibrium of* Z and it is characterized by $Z^\Sigma(q) = 0$. Any $q \in \Sigma^t$ is called a *tangential singularity* (or also *tangency point*) and it is characterized by $(X.f(q))(Y.f(q)) = 0$ (q is a tangent contact point between the trajectories of X and/or Y with Σ).

Consider $p \in \Sigma^t$. When the trajectory of X (resp., Y) by p belongs to Σ^+ (resp., Σ^-) we call it a *visible tangency*. When the trajectory of X (resp., Y) by p point belongs to Σ^- (resp.,

DOI: 10.5540/03.2015.003.01.0023

Σ^+) we call it an *invisible tangency*. A tangential singularity $p \in \Sigma^t$ is *singular* if p is a invisible tangency for both X and Y. On the other hand, a tangential singularity $p \in \Sigma^t$ is *regular* if it is not singular.

Let $W \in \chi$. Then we denote its flow by $\phi_W(t,p)$.

Definition 1. *The **local trajectory (orbit)** $\phi_Z(t,p)$ of a NSVF given by (1) is defined as follows:*

- *For $p \in \Sigma^+ \backslash \Sigma$ and $p \in \Sigma^- \backslash \Sigma$ the trajectory is given by $\phi_Z(t,p) = \phi_X(t,p)$ and $\phi_Z(t,p) = \phi_Y(t,p)$ respectively.*
- *For $p \in \Sigma^c$ such that $X.f(p) > 0$, $Y.f(p) > 0$ and taking the origin of time at p, the trajectory is defined as $\phi_Z(t,p) = \phi_Y(t,p)$ for $t \leq 0$ and $\phi_Z(t,p) = \phi_X(t,p)$ for $t \geq 0$. For the case $X.f(p) < 0$ and $Y.f(p) < 0$ the definition is the same reversing time.*
- *For $p \in \Sigma^e$ and taking the origin of time at p, the trajectory is defined as $\phi_Z(t,p) = \phi_{Z^\Sigma}(t,p)$ for $t \leq 0$ and $\phi_Z(t,p)$ is either $\phi_X(t,p)$ or $\phi_Y(t,p)$ or $\phi_{Z^\Sigma}(t,p)$ for $t \geq 0$. For the case $p \in \Sigma^s$ the definition is the same reversing time.*
- *For p a regular tangency point and taking the origin of time at p, the trajectory is defined as $\phi_Z(t,p) = \phi_1(t,p)$ for $t \leq 0$ and $\phi_Z(t,p) = \phi_2(t,p)$ for $t \geq 0$, where each ϕ_1, ϕ_2 is either ϕ_X or ϕ_Y or ϕ_{Z^Σ}.*
- *For p a singular tangency point $\phi_Z(t,p) = p$ for all $t \in \mathbb{R}$.*

Definition 2. *A **global trajectory (orbit)** $\Gamma_Z(t,p_0)$ of $Z \in \chi$ passing through p_0 is a union*

$$\Gamma_Z(t,p_0) = \bigcup_{i \in \mathbb{Z}} \{\sigma_i(t,p_i); t_i \leq t \leq t_{i+1}\}$$

*of preserving-orientation local trajectories $\sigma_i(t,p_i)$ satisfying $\sigma_i(t_{i+1},p_i) = \sigma_{i+1}(t_{i+1},p_{i+1}) = p_{i+1}$ and $t_i \to \pm\infty$ as $i \to \pm\infty$. A global trajectory is a **positive** (respectively, **negative**) **global trajectory** if $i \in \mathbb{N}$ (respectively, $-i \in \mathbb{N}$) and $t_0 = 0$.*

Definition 3. *Given $\Gamma_Z(t,p_0)$ a global trajectory, the set $\omega(\Gamma_Z(t,p_0)) = \{q \in V; \exists\,(t_n)$ satisfying $\Gamma_Z(t_n,p_0) \to q$ when $t \to +\infty\}$ (respectively $\alpha(\Gamma_Z(t,p_0)) = \{q \in V; \exists\,(t_n)$ satisfying $\Gamma_Z(t_n,p_0) \to q$ when $t \to -\infty\}$) is called **ω-limit** (respectively **α-limit**) **set of $\mathbf{\Gamma_Z(t,p_0)}$**. The ω-limit (respectively α-limit) set **of a point p** is the union of the ω-limit (respectively α-limit) sets of all global trajectories passing through p.*

Example 1. *Consider Figure 2. We observe that the global orbit passing through $q \in \Sigma$ is not necessarily unique. In fact, according to the third bullet of Definition 1, the positive local trajectory by the point $q \in \Sigma$ can provide three distinct paths, namely, Γ_1, Γ_2 and Γ_3. In particular, it is clear that the Existence and Uniqueness Theorem is not true in the scenario of NSVFs. Moreover, the ω-limit set of Γ_i, $i = 1,2,3$ is, respectively, a focus, a pseudo-equilibrium and a limit cycle and, consequently, the ω-limit set of q being the union of these objects is not a connected set. This fact is not predicted in the classical theory. Note that the α-limit set of q is a connected set composed by the pseudo-equilibrium p.*

Definition 4. *Consider $Z = (X,Y) \in \Omega$. A closed global orbit Δ of Z is a:*

(i) ***pseudo-cycle*** *if $\Delta \cap \Sigma \neq \emptyset$ and it does not contain neither equilibrium nor pseudo-equilibrium (See Figure 3).*

(ii) ***pseudo-graph*** *if $\Delta \cap \Sigma \neq \emptyset$ and it is a union of equilibria, pseudo equilibria and orbit-arcs of Z joining these points (See Figure 4).*

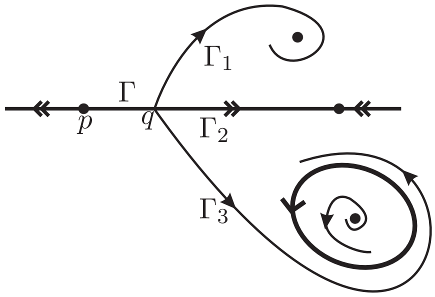

Figura 2: An orbit by a point is not necessarily unique.

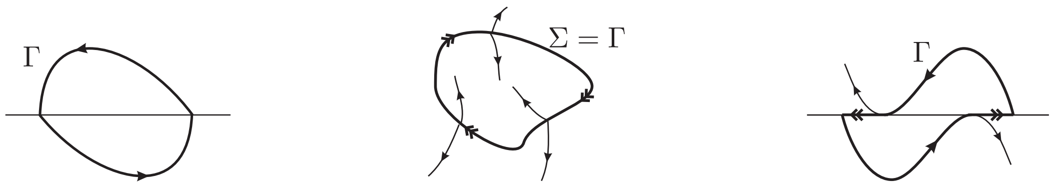

Figura 3: Possible kinds of pseudo-cycles.

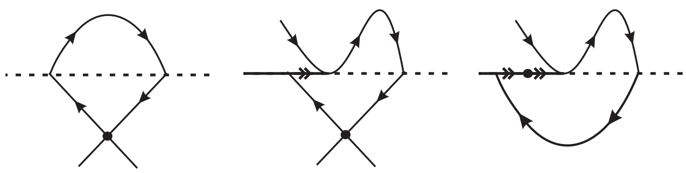

Figura 4: Examples of pseudo-graphs.

Definition 5. *A set $A \subset \mathbb{R}^2$ is* ***Z-invariant*** *if for each $p \in A$ and all global trajectory $\Gamma_Z(t, p)$ passing through p it holds $\Gamma_Z(t, p) \subset A$.*

Definition 6. *A set $M \subset \mathbb{R}^2$ is* ***minimal for $\mathbf{Z} \in \mathbf{\Omega}$*** *if*

(i) $M \neq \emptyset$;

(ii) M is compact;

(iii) M is Z-invariant;

(iv) M does not contain proper subset satisfying (i), (ii) and (iii).

Remark 1. *Observe that the pseudo-cycle Γ on the right of Figure 3 is the α-limit set of all global trajectories on a neighborhood of it, however Γ is not Z-invariant according to Definition 5. This phenomenon point out a distinct and amazing aspect not predicted for the classical theory about smooth vector fields where the α and ω-limit sets are invariant sets.*

3 Main Results

Theorem 1. *Let $Z = (X, Y) \in \Omega$. Assume that Z does not have sliding motion and it has a global trajectory $\Gamma_Z(t, p)$ whose positive trajectory $\Gamma_Z^+(t, p)$ is contained in a compact subset $K \subset V$. Suppose also that X and Y have a finite number of critical points in K, no one of them in Σ, and a finite number of tangency points with Σ. Then, the ω-limit set $\omega(\Gamma_Z(t, p))$ of $\Gamma_Z(t, p)$ is one of the following objects: (i) an equilibrium of X or Y; (ii) a periodic orbit of X or Y; (iii) a graph of X or Y; (iv) a pseudo-cycle; (v) a pseudo-graph; (vi) a singular tangency.*

As consequence, since the uniqueness of orbits and trajectories passing through a point is not achieved, we have the following corollary:

Corollary 1. *Under the same hypothesis of Theorem 1 the* ω*-limit set* $\omega(p)$ *of a point* $p \in V$ *is one of the objects described in items (i), (ii), (iii), (iv), (v) and (vi) or a union of them.*

The same holds for the α-limit set, reversing time. For the general case where sliding motion is allowed in Σ, we can not exhibit an analogous result. In fact, there exist *non-trivial minimal sets* (i.e., minimal sets distinct of an equilibrium point or of a closed trajectory) in this scenario (see Figure 5).

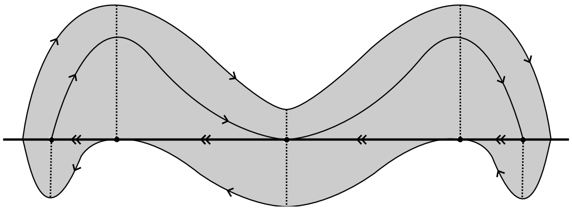

Figura 5: Non-trivial minimal set presenting non-empty interior.

Consider that $\Sigma = \Sigma^c \cup \Sigma^t$. In other words, Σ has only sewing and tangential points. We observe that the three first possibilities for the ω-limit set of $\Gamma_Z(t,p)$ in Theorem 1 are related with the classical Poincaré-Bendixson Theorem. Furthermore, the other possibilities appear due to the special type of discontinuous region Σ that we are considering (note that there are no escaping or sliding points on Σ). The proof of Theorem 1 takes into account the classical Poincaré-Bendixson Theorem and the concept of Poincaré return map for NSVFs.

Proof of Theorem 1. Consider $p \in V$. If there exists a time $t_0 > 0$ such that the global trajectory $\Gamma_Z(t,p)$ by p does not collide with Σ for $t > t_0$ then we can apply the classical Poincaré-Bendixson Theorem in order to conclude that one of the three first cases (i), (ii) or (iii) happens. Otherwise, there exists a sequence $(t_i) \subset \mathbb{R}$ of positive times, $t_i \to +\infty$, such that $p_i = \Gamma_Z(t_i,p) \in \Sigma$.

The hypothesis that we do not have sliding motion implies $Xf(p_i) \cdot Yf(p_i) \geq 0$. We observe that if $Xf(p_i) = 0$ and $Yf(p_i) \neq 0$ (resp., $Xf(p_i) \neq 0$ and $Yf(p_i) = 0$) then the trajectory of X (resp., Y) passing through p_i has an odd contact with Σ. For each $i \in \mathbb{N}$ we say that $p_i \in T(p)$ if one of the following cases happens: (i) $Xf(p_i) \cdot Yf(p_i) > 0$, (ii) $Xf(p_i) = 0$ and $Yf(p_i) \neq 0$, (iii) $Xf(p_i) \neq 0$ and $Yf(p_i) = 0$ or (iv) $Xf(p_i) = Yf(p_i) = 0$ and both have an odd contact order with Σ. If $Xf(p_i) = Yf(p_i) = 0$ and the contact order of both is even then we say that $p_i \in N(p)$. Observe that, by hypothesis, $N(p)$ is a finite set. We separate the proof in two cases: $T(p)$ is finite and $T(p)$ is not finite.

Assume that $T(p)$ is a finite set. We denote by n_p and t_p the number of elements of the sets $N(p)$ and $T(p)$ respectively. According to Definition 1, a global trajectory of Z by $p_l \in N(p)$ can follows one of two distinct paths. Let us denote by Γ_m an arc of $\Gamma_Z(t,p)$ connecting two consecutive points p_i and p_{i+1}, $i \in \mathbb{N}$. In this case there exists at most $2^{n_p} + t_p$ arcs Γ_m of $\Gamma_Z(t,p)$. So, there exists a (sub)set $\Upsilon \subset \{1, 2, \ldots, 2^{n_p} + t_p\}$ such that $\Gamma = \bigcup_{j \in \Upsilon} \Gamma_j$ is a closed orbit intersecting Σ (i.e., a pseudo-cycle) contained in $\Gamma_Z(t,p)$ and with the property that $\Gamma_Z(t,p)$ visit each arc Γ_j of Γ an infinite number of times. In what follows we prove that $\omega(\Gamma_Z(t,p)) = \Gamma$. In fact, as $\Gamma_Z(t,p)$ must visit each arc Γ_j of Γ an infinite number of times then $\Gamma \subset \omega(\Gamma_Z(t,p))$. On the other hand, if $x_0 \in \omega(\Gamma_Z(t,p))$ then there exists a sequence $(s_k) \subset \mathbb{R}$, $s_k \to +\infty$, such that $\Gamma_Z(s_k,p) = x_k \to x_0$. Moreover, since $\Gamma_Z(t,p)$ also is composed by a finite number of arcs Γ_m, $s_k \to +\infty$ and $\Gamma_Z(t,p)$ has no equilibria (otherwise it does not visit Σ infinitely many times), there exists a subsequence (x_{k_j}) of (x_k) that visits some arcs Γ_m infinitely many times. Since Γ includes all arcs Γ_j for which the global trajectory visit Γ_j for an infinite sequence of times, $x_{k_j} \in \Gamma$ a compact set, and consequently $x_0 \in \Gamma$.

DOI: 10.5540/03.2015.003.01.0023

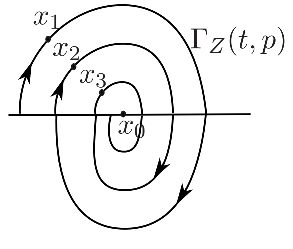

Figura 6: Case where there exists a singular tangency in $\omega(\Gamma_Z(t,p)) \cap \Sigma$.

Now assume that $T(p)$ in not a finite set. In this case, there exist a point $q \in \Sigma$ and a subsequence $(t_{i_j}) = (s_j)$ of (t_i) such that

$$\lim_{j\to\infty} \Gamma_Z(s_j,p) = q \tag{2}$$

since $\Gamma_Z^+(t,p) \subset K$, a compact set. Observe that $q \in \omega(\Gamma_Z(t,p)) \cap \Sigma \neq \emptyset$. As we do not have sliding motion, for each $x \in \omega(\Gamma_Z(t,p)) \cap \Sigma$, we have only two options for it: either x is a singular tangency or x is a regular point.

If there exists $x_0 \in \omega(\Gamma_Z(t,p)) \cap \Sigma$ a singular tangency then $\omega(\Gamma_Z(t,p)) = \{x_0\}$. In fact, when both X and Y have an invisible tangency point at x_0 and there exists a sequence $(s_k) \subset \mathbb{R}$, $s_k \to +\infty$, such that $\Gamma_Z(s_k,p) = x_k \to x_0$ then there is a small neighborhood V_{x_0} of x_0 in V such that all trajectory of Z that starts at a point of V_{x_0} converges to x_0. See Figure 6. Therefore, $\omega(\Gamma_Z(t,p)) = \{x_0\}$ and $x_0 = q$.

Suppose now that all points in $\omega(\Gamma_Z(t,p)) \cap \Sigma$ are regular ones. Again we separate the analysis in two cases: either $\omega(\Gamma_Z(t,p))$ contains equilibria or contains no equilibria. Consider the case when $\omega(\Gamma_Z(t,p))$ contains no equilibria. Let q as in Equation (2). If $q \in T(q)$ then it is clear that the local trajectory passing through q is unique and $\Gamma_Z(\varepsilon,q) \in \omega(\Gamma_Z(t,p))$ for $\varepsilon > 0$ sufficiently small. If $q \in N(q)$ then, since q can not be a singular tangency, q is a visible tangency for both X and Y. So, there are two possible choices for the positive local trajectory of Z passing through q and at least one of them is such that it is contained in $\omega(\Gamma_Z(t,p))$. By continuity, the global trajectory $\Gamma(t,q)$ of Z that passes through q, contained in $\omega(\Gamma_Z(t,p))$, must come back to a neighborhood V_q of q in Σ. The late affirmation is true, because if it does not come back then it remains in Σ^+ or in Σ^-. So, the set $\omega(\Gamma(t,q))$ is a periodic orbit of X or Y, because there are no singular points in $\omega(\Gamma_Z(t,p))$. But it is a contradiction with the fact that the orbit $\Gamma_Z(t,p)$ must visit any neighborhood of q infinitely many times. Moreover, by the Jordan Curve Theorem, $\Gamma(t,q) \cap V_q = \{q\}$, otherwise there exists a flow box not containing q for which $\Gamma(t,q)$ and, consequently, $\Gamma(t,p)$, do not depart it. This is a contradiction with the fact that the orbit $\Gamma_Z(t,p)$ must visit any neighborhood of q infinite many times. Therefore, $\Gamma_Z(t,q)$ is closed (i.e., is a pseudo-cycle) and $\omega(\Gamma_Z(t,p)) = \Gamma_Z(t,q)$.

The remaining case is when $\omega(\Gamma_Z(t,p))$ has equilibria either of X or of Y. In this case for each regular point $q \in \omega(\Gamma_Z(t,p))$ consider the local orbit $\Gamma_Z(t,q)$ which is contained in $\omega(\Gamma_Z(t,p))$. The set $\omega(\Gamma_Z(t,q))$ can not be a periodic orbit or a graph contained in Σ^+ or in Σ^-, because the orbit $\Gamma_Z(t,p)$ must visit any neighborhood of q infinite many times. So, the unique option is that $\omega(\Gamma_Z(t,q)) = \{z_i\}$ where z_i is an equilibrium of X or of Y. Similarly, the α-limit set $\alpha(\Gamma_Z(t,q)) = \{z_j\}$ where z_j is an equilibrium of X or of Y. Thus, with an appropriate ordering of the equilibria z_k, $k = 1, 2 \ldots, m$, (which may not be distinct) and regular orbits $\Gamma_k \subset \omega(\Gamma_Z(t,p))$, $k = 1, 2 \ldots, m$, we have

$$\alpha(\Gamma_k) = z_k \quad \text{and} \quad \omega(\Gamma_k) = z_{k+1}$$

for $k = 1, \ldots, m$, where $z_{m+1} = z_1$. It follows that the global trajectory $\Gamma_Z(t,p)$ either spirals down to or out toward $\omega(\Gamma_Z(t,p))$ as $t \to +\infty$. It means that in this case $\omega(\Gamma_Z(t,p))$ is a pseudo-graph composed by the equilibria z_k and the arcs Γ_k connecting them, $k = 1, \ldots, m$.

This concludes the proof of Theorem 1. □

Now we perform the proof of Corollary 1. In Example 2 below we illustrate its consequences.

Proof of Corollary 1. In fact, since by Definition 3 the ω-limit set of a point is the union of the ω-limit set of all global trajectories passing through it, the conclusion is obvious. □

Example 2. *Consider Figure 7. Here we observe a NSVF without sliding motion on Σ where the conclusions of Theorem 1 and Corollary 1 can be observed. Since the uniqueness of trajectories by p is not achieved (neither for positive nor for negative times) both the α and the ω-limit sets are disconnected sets. The α-limit set of p is composed by the focus α_1 and the s-singular tangency point α_2. The ω-limit set of p is composed by the saddle ω_1 and the periodic orbit Γ_1.*

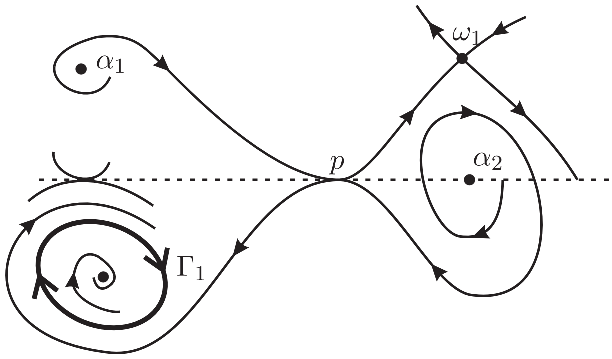

Figura 7: Both the α-limit set $\{\alpha_1, \alpha_2\}$ and the ω-limit set $\{\omega_1, \Gamma_1\}$ of the point p are disconnected. Sliding motion on Σ is not allowed.

Acknowledgments. The first author is partially supported by FAPESP-BRAZIL grant 2007/06896-5. The second author is partially supported by FAPESP-BRAZIL grant 2012/00481-6. The third author is supported by the FAPESP-BRAZIL grants 2010/18015-6 and 2012/05635-1

Referências

[1] M. di Bernardo, C.J. Budd, A.R. Champneys and P. Kowalczyk, *Piecewise-smooth Dynamical Systems − Theory and Applications*, Springer-Verlag (2008).

[2] A.F. Filippov, *Differential Equations with Discontinuous Righthand Sides*, Mathematics and its Applications (Soviet Series), Kluwer Academic Publishers-Dordrecht, 1988.

[3] J. Sotomayor, A.L. Machado, *Structurally stable discontinuous vector fields on the plane*, Qual. Theory Dyn. Syst., **3** (2002), 227–250.

[4] M.A. Teixeira, *Perturbation Theory for Non-smooth Systems*, Meyers: Encyclopedia of Complexity and Systems Science **152** (2008).

