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On certain shallow water models, scaling invariance and strict self-adjointness

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Resumo: *In this work we establish conditions for a class of third order partial differential equations to be strictly self-adjoint and scale invariant. The obtained family of equations includes the Benjamin-Bona-Mahony, Camassa-Holm and Novikov equations. Using the strict self-adjointness and Ibragimov's conservation theorem, we establish some local conservation laws for some of the mentioned equations.*

Palavras-chave: *Strict self-adjointness, Ibragimov's conservation theorem, conservation laws.*

1 Historical survey

During the last century, a sequence of papers, starting with [27], showed and enlightened many properties of the well known Korteweg – de Vries equation

$$u_t = u_{xxx} + uu_x. \tag{1}$$

Later, in [1], a new equation called Benjamin-Bona-Mahoney (BBM), given by

$$u_t = u_{txx} + uu_x, \tag{2}$$

was derived as an "alternative" for the KdV.

Although equation (2) was derived using the same formal justification for obtaining (1), the differences between both equations are greater than the fact that (1) is an evolution equation whereas (2) is not. In [1] the authors found three conserved quantities on the solutions of (2). Later in [30], those conservation laws obtained were proved to be the only three admitted by (2). This fact shows a dramatic difference between BBM and KdV since the last one admits an infinite number of conserved quantities [28].

More recently, Camassa and Holm [5] using Hamiltonian methods derived the famous Camassa-Holm (CH) equation

$$u_t - u_{txx} + 3uu_x = 2u_xu_{xx} + uu_{xxx}. \tag{3}$$

The last equation possesses remarkable properties such as peakon solutions and a bi-hamiltoninan structure, see [5], which implies in the existence of an infinite number of conserved quantities, just like the KdV equation [15, 28, 26].

Since then, a considerable number of papers have been dedicated to derive third order non-evolutionary dispersive equations having similar properties as those known to KdV and CH equation. To cite a few number of examples, it was derived in [8] an integrable equation having peakon solutions with first order nonlinearities, while in [9] another integrable equation, combining linear dispersion such as the KdV equation and a nonlinear dispersion like the CH equation, was discovered. In [8] an integrable equation with peakon solutions was considered and, more recently, Novikov [29] has discovered the equation

$$u_t - u_{txx} + 4u^2u_x = 3uu_xu_{xx} + u^2u_{xxx}, \tag{4}$$

which not only admits peakon solutions and cubic nonlinearities, but it is also integrable [17].

In [28] it was shown that the KdV equation possesses infinitely many conservation laws. This was the start point of a considerable number of papers dealing with the properties of a certain equation and the existence of an infinite number of conserved quantities.

Noether theorem showed a deeper and closer relation between symmetries and conservation laws for the Euler-Lagrange equations. She showed that for each conservation law of a differential equation, or system, there is a symmetry property related to it. However, Noether theorem requires that the equation is an Euler-Lagrange equation, and although the KdV equation is not an Euler-Lagrange equation, it can be transformed in one using the differential substitution $u = v_x$. Then, from it, one arrives at the equation $v_{tx} = v_{xxxx} + v_xv_{xx}$, which is an Euler-Lagrange equation.

The first paper relating symmetries (not necessarily Lie point symmetries) of the KdV equation and an infinite number of local conservation laws for it was [21], in which Ibragimov showed how to construct local conservation laws using symmetries other than the Lie point symmetries.

In order to construct the conserved vectors, Ibragimov first established a non-local conserved vector. Then he showed that the KdV equation is strictly self-adjoint [20, 21, 25] and, consequently, the non-local conserved quantities can be transformed in locals one. A considerable number of integrable equations has a common property: strict self-adjointness.

Ibragimov in [21] showed that KdV is strictly self-adjoint. In [24] it was shown that the CH equation has also the same property, as well as in [4] it was proved that the Novikov equation is strictly self-adjoint. In particular, with respect to (3) and (4), the obtained results in [6], [24] and [4] shows some common facts:

1. both equations are strictly self-adjoint;
2. both equations admit the scaling symmetry $(x, t, u) \mapsto (x, \lambda^{-b}t, \lambda u)$, for a certain value of b, whose corresponding generator is

$$X_b = u\frac{\partial}{\partial u} - bt\frac{\partial}{\partial t}; \tag{5}$$

Since Ibragimov's concepts on self-adjointness have been introduced, a considerable number of papers has been dealing with the problem of finding classes of differential equations with some self-adjoint property, see, for instance, [10, 11, 12, 13, 14, 16, 32].

Therefore, motivated by those recent results and provoked by the classification carried out in [29], in which certain generalizations of the CH equation possessing infinite hierarchies of higher symmetries were considered, we tried to determine which conditions are necessary and sufficient for the equation

$$F = u_t + \epsilon u_{txx} + f(u)u_x + g(u)u_xu_{xx} + h(u)u_{xxx} = 0 \tag{6}$$

to be strictly self-adjoint. After that we restrict ourselves to find a subclass of (6) that admits a certain scaling symmetry. Then we can find local conservation laws using the conservation theorem Ibragimov proposed in recent years. Next sections will talk about theory and original results obtained. All results presented here can be found more detailed in [7].

2 Strict self-adjointness and invariance

According to Ibragimov [21, 22, 23, 25], a differential equation

$$F(x, u, u_{(1)}, \dots, u_{(n)}) = 0 \tag{7}$$

is said to be strictly self-adjoint if, and only if,

$$F^*\Big|_{v=u} = \lambda F, \tag{8}$$

for some differential function λ, where $x \in \mathbb{R}^n$, $u_{(i)}$ denotes the derivatives of u of order i, and F^* is the adjoint equation:

$$F^* := \frac{\delta}{\delta u}(vF),$$

where the Euler Lagrange operator $\frac{\delta}{\delta u}$ is given by the formal sum

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} + \sum_{j=1}^{\infty}(-1)^j D_{i_1}\dots D_{i_j}\frac{\partial}{\partial u_{i_1\dots i_j}}.$$

Theorem 2.1. *Equation* (6) *is strictly self-adjoint if and only if*

$$g(u) = \frac{(uh)'}{u} + \frac{c}{u},$$

where c is an arbitrary constant.

To prove the theorem we have to first find the adjoint equation $F^* = 0$, which is given by

$$F^* = v[g'(u)u_x u_{xx} + f'(u)u_x + h'(u)u_{xxx}] - D_t(v) - D_x[v(f(u) + g(u)u_{xx})] +$$

$$+ D_x^2[vg(u)u_x] - D_x^2 D_t(\epsilon v) - D_x^3[vh(u)].$$

Using condition (8), it is found that $\lambda = -1$ and the following constraints, arising from the coefficients of u_x^3 and $u_x u_{xx}$, respectively:

$$\begin{aligned} (ug)'' - (uh)''' &= 0 \\ (ug)' - (uh)'' &= 0. \end{aligned} \tag{9}$$

Clearly the last condition implies the first one and integrating it once, we obtain the desired result.

We can now restrict ourselves to find the subfamily of strictly self-adjoint equations admitting the scaling symmetry $(x, t, u) \mapsto (x, \lambda^{-b}t, \lambda u)$.

We recall that a differential equation (7) admits a symmetry $(x, u) \mapsto (\hat{x}, \hat{u})$, $x \in \mathbb{R}^n$ if its corresponding infinitesimal generator

$$X = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}$$

satisfies the condition

$$X^{(n)}F = \sigma F,$$

where $X^{(n)}$ is the n-th prolongation of X, see [2, 3, 18, 19, 31], and σ is a suitable differential function.

Using Theorem 2.1 and the invariance condition $X^{(3)}F = \sigma F$, we obtain explicit conditions for funtions $f(u), g(u)$ and $h(u)$ by having the following four-parameter family of scale invariant strictly self-adjoint equations

$$u_t + \epsilon u_{txx} + \gamma u^b u_x = (1 + b)\beta u^{b-1}u_x u_{xx} + \beta u^b u_{xxx}, \tag{10}$$

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which includes equations (2), (3) and (4). Moreover, taking $b = 1$, $\epsilon = -\beta = \alpha^2$ and $\gamma = 3$, we arrive, up to a translation $u \mapsto u + u_0/\alpha^2$, at the Dullin-Gotwald-Holm equation

$$u_t - \alpha^2 u_{txx} + 3uu_x = \alpha^2(uu_{xxx} + 2u_x u_{xx}) + u_0 u_{xxx}, \tag{11}$$

which is also integrable, see [9]. The term u_0 corresponds to the coefficient of the linear dispersion of the equation and when $u_0 \to 0$ and $\alpha = 1$, such equation turns back to the CH equation. However, if $u_0 \neq 0$, (11) does not admit the generator (5). For instance, when $u_0 = 1$ and $\epsilon \to 0$, one easily obtains the KdV equation, and it is not scale invariant. We also observe that at the limit of the dispersionless u_0, $\alpha \to 0$, equation (11) is reduced to the Riemann equation $u_t + 3uu_x = 0$. More generally, when the dispersion effects are neglected in (10), that is, ϵ, $\beta \to 0$, one obtains a family of Riemann equations given by

$$u_t + \gamma u^b u_x = 0. \tag{12}$$

3 Conservation laws

Ibragimov in [20, 21, 22, 23, 25] established connections between a system of differential equations, formed by a differential equation $F = 0$ and its adjoint $F^* = 0$, and the theorem proved by Noether. He showed that a symmetry admitted by an equation is inherited by its adjoint equation. Moreover, he proved that this symmetry is a variational symmetry for the system, with the Lagrangean $\mathcal{L} = vF$, and then one can find non-local conservation laws using Noether theorem.

For this particular case, Noether theorem states thay

$$\begin{aligned} C^0 &= \tau\mathcal{L} + W\left[\frac{\partial\mathcal{L}}{\partial u_t} + D_x^2\left(\frac{\partial\mathcal{L}}{\partial u_{txx}}\right)\right] - D_x(W)D_x\left(\frac{\partial\mathcal{L}}{\partial u_{txx}}\right) + D_x^2(W)\frac{\partial\mathcal{L}}{\partial u_{txx}}, \\ C^1 &= \xi\mathcal{L} + W\left[\frac{\partial\mathcal{L}}{\partial u_x} - D_x\left(\frac{\partial\mathcal{L}}{\partial u_{xx}}\right) + D_x^2\left(\frac{\partial\mathcal{L}}{\partial u_{xxx}}\right) + D_x D_t\left(\frac{\partial\mathcal{L}}{\partial u_{xxt}}\right) + D_t D_x\left(\frac{\partial\mathcal{L}}{\partial u_{xtx}}\right)\right] \\ & D_x(W)D_x\left[\frac{\partial\mathcal{L}}{\partial u_{xx}} - D_x\left(\frac{\partial\mathcal{L}}{\partial u_{xxx}}\right) - D_t\left(\frac{\partial\mathcal{L}}{\partial u_{xxt}}\right)\right] - D_t(W)D_x\left(\frac{\partial\mathcal{L}}{\partial u_{xtx}}\right) + D_x^2(W)\frac{\partial\mathcal{L}}{\partial u_{xxx}} \\ & + D_t D_x(W)\frac{\partial\mathcal{L}}{\partial u_{xtx}} + D_x D_t(W)\frac{\partial\mathcal{L}}{\partial u_{xxt}}, \end{aligned} \tag{13}$$

provides a conserved current for our equation, where $W = \eta - \xi u_x - \tau u_t$ and the formal Lagrangian $\mathcal{L}$ is given by

$$\mathcal{L} = v\left[u_t + \epsilon\frac{u_{txx} + u_{xtx} + u_{xxt}}{3} + f(u)u_x + g(u)u_x u_{xx} + h(u)u_{xxx}\right]. \tag{14}$$

With the concept of strict self-adjointness, it is possible to remove the non-local variable v that arises from the definition of the Lagrangean $\mathcal{L}$ by setting $v = u$, see [23, 25]. That being, the adjoint equation is then equivalent to the original equation and Noether theorem provides local conservation laws for the equation initially considered. There are generalizations of this strict self-adjoitness concept, but here are of no interest right now to our studies presented here.

For the family (10) of strictly self-adjoint equations, the scale symmetry generator (5) yields the conserved current of components

$$C^0 = u^2 - \epsilon u_x^2, \quad C^1 = \frac{2}{2+b}\gamma u^{b+2} - 2\beta u^{b+1}u_{xx} + 2\epsilon u u_{tx}. \tag{15}$$

The conserved density C^0 of (15) is valid for all values of b, while the conserved flux C^1 corresponds to the cases when $b \neq -2, 0$.

Also, the conservation law of components given by (15) includes quantities known in the literature, coming from generator (5), for equations (2), (3) and (4). The conserved vector (15) also provides a known conservation law for the class of Riemann equations (12), but only if $b \neq -2$.

In [7], we also found the conservation currents for cases $b = 0$ and $b = -2$. The component C^0 remains unchanged, while the component C^1, for $b = 0$ and $b = -2$, is given respectively by

$$C_0^1 = \gamma u^2 - 2(\beta - c)uu_{xx} + 2\epsilon uu_{tx} - cu_x^2,$$

$$C_{-2}^1 = -2\beta \frac{u_{xx}}{u} + 2\gamma \ln u + 2\epsilon uu_{xt}.$$

That way, for all values of b one can obtain the conservation law arising from a scale symmetry for equation (10), a class of equations that contains many of the most famous shallow water models studied.

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