Proceeding Series of the Brazilian Society of Applied and Computational Mathematics, Vol. 3, N. 1, 2015.
Trabalho apresentado no XXXV CNMAC, Natal-RN, 2014.

The boundary of a class of Rauzy fractals

Jefferson L. Bastos
Depto de Matemática, IBILCE, UNESP,
15054-000, São José do Rio Preto, SP
E-mail: jeferson@ibilce.unesp.br

Tatiana Miguel Rodrigues
Depto de Matemática, FC, UNESP
17033-360, Bauru, SP
E-mail: tatimi@fc.unesp.br.

Resumo: *In this work we give arithmetical properties for the boundary of a class of Rauzy fractals R_a given by the polynomial $x^3 - ax^2 + x - 1,\ a \geq 3$. We give an automaton that generates this boundary and we prove that it is homeomorphic to S^1.*

Palavras-chave: *Rauzy fractal, Tiling, Automaton*

1 Introduction

The Rauzy fractal was studied by many mathematicians and was connected to to many topics as: numeration systems ([8],[6]), geometrical representation of symbolic dynamical system ([2], [7]), multidimensional continued fractions and simultaneous approximations ([3], [5]), auto-similar tilings ([2], [8]) and Markov partitions of Hyperbolic automorphisms of Torus ([7], [8]). There are many ways of constructing Rauzy's fractals one of them is by β-expansions.

Let $\beta > 1$ be a fixed real number and x any positive real number. Using Greedy algorithm we can write $x = \sum_{i=N_0}^{\infty} a_{-i}\beta^{-i},\ a_{-i} \in \mathbb{Z} \cap [0, \beta)$ (β expansion of x). A Pisot number $\beta > 1$ is an algebraic integer whose conjugates other than itself have modulus less than one. Let $Fin(\beta)$ be a set consisting of all finite $\beta-$expansion and consider the condition

$$Fin(\beta) = \mathbb{Z}[\beta^{-1}] \cap \mathbb{R}^+ \qquad \text{(property } F\text{)}$$

The Pisot numbers that satisfy property (F) were characterized in [1] as being exactly the set of dominant roots of the polynomial (with integers coefficients)

$$P_{a,b}(x) = x^3 - ax^2 - bx - 1,\ a \geq 0,\ -1 \leq b \leq a + 1.$$

(If $b = -1$ add the restriction $a \geq 2$). In particular, this set is divided into three subsets:

a) $0 \geq b \geq a$, and in this case $d(1, \beta) = \cdot ab1$.

b) $b = -1,\ a \geq 2$. In this case $d(1, \beta) = \cdot(a - 1)(a - 1)01$.

c) $b = a+1$, and in this case $d(1, \beta) = \cdot(a+1)00a1$, where $d(1, \beta)$ is the Rényi β-representation of 1 (see citerényi).

We can associated a fractal to each of this cases above. The fractal associated to (b) is given by

$$\mathcal{R}_a = \left\{ \sum_{i=2}^{\infty} a_i\alpha^i,\ a_i a_{i-1} a_{i-2} a_{i-3} <_{lex} (a - 1)(a - 1)01,\ \forall i \geq 5 \right\},$$

where $<_{lex}$ is the lexicographic order on finite words. In [4] we prove some topological and arithmetic properties of $\mathcal{R}_a$ and give a complete description of the boundary of $\mathcal{R}_2$. The purpose of this work is to present a complete description of the boundary of $\mathcal{R}_a, a \geq 3$. For this we need the following results:

Theorem 1.1. *R_a induces a periodic tiling of the plane $\mathbb{C}$ modulo $\mathbb{Z}(\alpha-1)+\mathbb{Z}(\alpha^2-\alpha)$. Moreover*

$$\partial R_a = \bigcup_{v\in B}(R_a \cap (R_a+v)),\ B = \{\pm(\alpha-1); \pm(\alpha^2-\alpha); \pm(\alpha^2-1); \pm(\alpha^2-2\alpha+1)\},$$

$R_a \cap (R_a + (\alpha^2-1)) = \{-1\};\ R_a \cap (R_a + (\alpha^2-2\alpha+1)) = \{-\alpha\}.$

Theorem 1.2. *Consider $g(z) = \alpha - 1 + \alpha(z)$. Then*

$$R_{\alpha-1} = R_a \cap (R_a + \alpha - 1) = g(R_a \cap (R_a + (\alpha^2-\alpha)))$$

2 Parametrization of the boundary of $\mathcal{R}_a, \forall a \geq 3$

In this section we give a complete description of the boundary of R_a. By theorem 1.1 we have to study the sets $R_a \cap (R_a + v),\ v \in \{\pm(\alpha-1), \pm(\alpha^2-\alpha)\}$. By symmetry and theorem 1.2 we can just study the set $R_{\alpha-1}$.

In [4] we show that the automaton below characterize the boundary of $\mathcal{R}_a$.

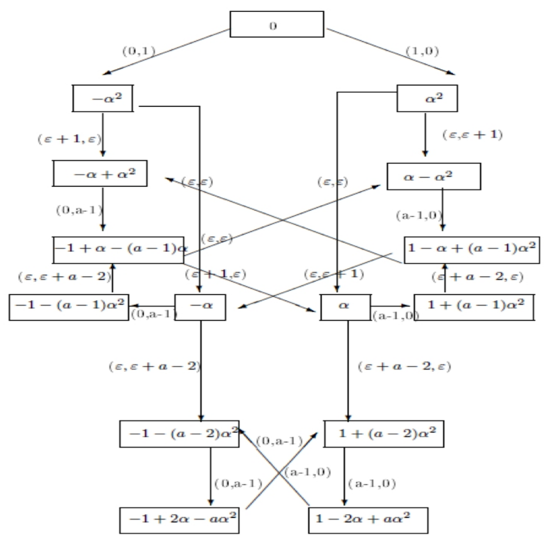

Figure 1: Automaton $\mathcal{A}$

Using the automaton we have $R_{\alpha-1} = R^1_{\alpha-1} \bigcup R^2_{\alpha-1}$ where
$R^1_{\alpha-1} = \{z = \alpha - 1 + \sum_{i=3} a_i\alpha^i = (a-1)\alpha^2 + \sum_{i=3} b_i\alpha^i, (a_3, b_3) = (\epsilon, \epsilon), \epsilon = 0, ..., a-1\}$,
$R^2_{\alpha-1} = \{z = \alpha - 1 + \sum_{i=3} a_i\alpha^i = (a-1)\alpha^2 + \sum_{i=3} b_i\alpha^i, (a_3, b_3) = (\epsilon+1, \epsilon), \epsilon = 0, ..., a-2\}$.

Lemma 2.1. *Considering* $R^{1,t}_{\alpha-1} = \{z \in R^1_{\alpha-1}; (a_3, b_3) = (t, t),\ t = 0, 1...a-1\}$, $R^{2,t}_{\alpha-1} = \{z \in R^2_{\alpha-1}; (a_3, b_3) = (t, t-1),\ t = 1, 2...a-1\}$ *and* $R'_{\alpha-1} = \{z \in R_{\alpha-1}; a_3 \neq a-1\}$. *We have:*

1. $g_{2k+1} : R_{\alpha-1} \longrightarrow R^{2,a-1-k}_{\alpha-1}$ *given by* $g_{2k+1}(z) = -1 - k\alpha^3 + \alpha^3 z,\ k = 0, ..., a-2$ *is bijective.*

2. $g_{2(a-1)} : R_{\alpha-1} \longrightarrow R^{1,0}_{\alpha-1}$ *given by* $g_{2(a-1)}(z) = \alpha - 1 + \alpha^2 z$*is bijective.*

3. $g_{2k} : R'_{\alpha-1} \longrightarrow R^{1,a-1-k}_{\alpha-1}$ *given by* $g_{2k}(z) = \alpha - 1 + (a-1-k)\alpha^3 + \alpha^2 z, k = 0, ..., a-2$ *is bijective.*

DOI: 10.5540/03.2015.003.01.0013

Corollary 2.2.

$$R_{\alpha-1} = \bigcup_{i=0}^{2(a-1)} g_i(X)$$

where $X = R_{\alpha-1}$ if i is odd and $2(a-1)$ and $X = R'_{\alpha-1}$ if i is even.

Using the previous notation and taking $u = -1, v = -(a-1)\alpha - \alpha^{-1}, w = -1 - \alpha^3$ we have the following lemma.

Lemma 2.3. $g_{2k}(-1-\alpha^3) = -1-\alpha^2-k\alpha^3-(a-1)\alpha^4 = g_{2k+1}(-(a-1)\alpha-\alpha^{-1}), k = 0, ..., a-2,$ *and* $g_{2k+1}(-1) = -1-(k+1)\alpha^3 = g_{2(k+1)}(-(a-1)\alpha-\alpha^{-1}), k = 0, ..., a-2.$

Lemma 2.4. *Take $r = 2a-1$. Then we have*

1. $\lim_{n\longrightarrow\infty} (g_0 \circ g_{r-1})^n(z) = u = -1.$
2. $\lim_{n\longrightarrow\infty} (g_{r-1} \circ g_0)^n(z) = v = -(a-1)\alpha - \alpha^{-1}, \ \forall z \in R'_{\alpha-1}.$

Lemma 2.5. *Take $t \in [0,1], a \geq 3, r = 2a-1$. Then*

1. $t = \frac{a_1}{r} + B + \sum_{\substack{k=3 \\ m_k+n_k=k}}^{\infty} \frac{a_k}{(r-2)^{m_k} r^{n_k}}$ *where*

   (a) $B = \begin{cases} \frac{a_2}{r^2} & , \ a_2 \in \{0,1,2,...,r-1\} \text{ if } a_1 \in \{1,3,5,...,r-2,r-1\} \\ \frac{a_2}{(r-2)r} & , \ a_2 \in \{0^*,1^*,...,(r-3)^*\} \text{ if } a_1 \in \{0,2,4,...,r-3\} \end{cases}$

   and for $i \geq 3$ we have:

   (b) *If $a_{i-1} = 0, 0^*$, $i-1$ even or $a_{i-1} = (r-3)^*, r-1$, $i-1$ odd or $a_{i-1} = r-2, (2n-1), (2n-1)^*, n = 1, ..., a-2$ then $a_i \in \{0,1,2...,r-1\}$, $m_i = m_{i-1}$ and $n_i = n_{i-1}+1$.*

   (c) *If $a_{i-1} = 0, 0^*$, $i-1$ odd or $a_{i-1} = (r-3)^*, r-1$, $i-1$ even or $a_{i-1} = r-3, (2n), (2n)^*, n = 1, ..., a-3$ then $a_i \in \{0^*, 1^*, ...(r-3)^*\}$, $m_i = m_{i-1}+1$ and $n_i = n_{i-1}$.*

2. *If $|t'-t| < (r-2)^m r^n$ with $m+n = N$ then there exists $k < N$ such that*

   (a) $t = a_1 \ldots a_{k-1} a_k (r-1)(r-1)(r-3)^*(r-1)(r-3)^* ... a_{N+1} ...,$ $t' = a_1 \ldots a_{k-1}(a_k + 1)00...a'_{N+1}...$ *if k is even and $a_k = 0$, $0^\star, r-2, (2n-1), (2n-1)^*, n = 1, ..., a-2$.*

   (b) $t = a_1 \ldots a_{k-1} a_k (r-1)(r-3)^*(r-1)(r-3)^* ... a_{N+1} ...,$ $t' = a_1 \ldots a_{k-1}(a_k+1)00...a'_{N+1}...$ *if k is odd and $a_k = r-2, (2n-1), (2n-1)^*, n = 1, ..., a-2$.*

   (c) $t = a_1 \ldots a_{k-1} a_k (r-3)^*(r-3)^*(r-1)(r-3)^*(r-1) ... a_{N+1} ...,$ $t' = a_1 \ldots a_{k-1}(a_k + 1)00...a'_{N+1}...$ *if k is odd and $a_k = 0$, $0^\star, r-3, (2n), (2n)^*, n = 1, ..., a-3$.*

   (d) $t = a_1 \ldots a_{k-1} a_k (r-3)^*(r-1)(r-3)^*(r-1) ... a_{N+1} ...,$ $t' = a_1 \ldots a_{k-1}(a_k+1)00...a'_{N+1}...$ *if k is even and $a_k = r-3, (2n), (2n)^*, n = 1, ..., a-3$.*

Proof:

1. Take $t, t' \in [0,1]$, $|t'-t| < (r-2)^m r^n$ with $m+n = N$ and suppose $t < t'$. Then exist $k \in \mathbb{N}$, $k < N$ such that $t = a_1 \ldots a_{k-1} a_k a_{k+1} ... a_N a_{N+1} ...$, $t' = a_1 \ldots a_{k-1} a'_k a'_{k+1} ... a'_N a'_{N+1} ...$, $a_k < a'_k$ and

$$t' - t = \frac{(a'_k - a_k)}{(r-2)^{m_k} r^{n_k}} + ... = \frac{(a'_k - a_k - 1)}{(r-2)^{m_k} r^{n_k}} + \frac{1}{(r-2)^{m_k} r^{n_k}} + ...$$

As $m_k + n_k = k$ and $|t'-t| < (r-2)^m r^n, m+n > N > k$ then $a'_k - a_k - 1 = 0$, that is, $a'_k = a_k + 1$.

DOI: 10.5540/03.2015.003.01.0013

(a) Let k be an even number and $a_k = 0, 0^*, r-2, (2n-1), (2n-1)^*, n = 1, ..., a-2$. Then $a_{k+1} \in \{0, 1, ..., r-1\}$ and we can write

$$\frac{1}{(r-2)^{m_k} r^{n_k}} = \frac{r-1}{(r-2)^{m_k} r^{n_k+1}} + \sum_{i=0}^{\infty} \left(\frac{r-1}{(r-2)^{m_k+i} r^{n_k+2+i}} + \frac{(r-3)^*}{(r-2)^{m_k+1+i} r^{n_k+2+i}} \right).$$

Therefore

$$t' - t = \frac{a'_{k+1}}{(r-2)^{m'_{k+1}} r^{n'_{k+1}}} - \frac{a_{k+1}}{(r-2)^{m_k} r^{n_k+1}} + \frac{(r-1)}{(r-2)^{m_k} r^{n_k+1}} + ... =$$

$$= \frac{a'_{k+1}}{(r-2)^{m'_{k+1}} r^{n'_{k+1}}} + \frac{r-1-a_{k+1}}{(r-2)^{m_k} r^{n_k+1}} + ...$$

where $m'_{k+1} + n'_{k+1} = m_k + n_k + 1 = k+1$. As $|t'-t| < (r-2)^m r^n, m+n > N \geq k+1$ then $\frac{a'_{k+1}}{(r-2)^{m'_{k+1}} r^{n'_{k+1}}} + \frac{r-1-a_{k+1}}{(r-2)^{m_k} r^{n_k+1}} = 0$ and it is possible only with $a'_{k+1} = 0, 0^*$ and $a_{k+1} = r-1$.
As $a_{k+1} = r-1$ and $k+1$ is an odd number, then $a_{k+2} \in \{0, 1, ..., r-1\}$ we have

$$t' - t = \frac{a'_{k+2}}{(r-2)^{m'_{k+2}} r^{n'_{k+2}}} - \frac{a_{k+2}}{(r-2)^{m_k} r^{n_k+2}} + \frac{r-1}{(r-2)^{m_k} r^{n_k+2}} + ... =$$

$$= \frac{a'_{k+2}}{(r-2)^{m'_{k+2}} r^{n'_{k+2}}} + \frac{r-1-a_{k+2}}{(r-2)^{m_k} r^{n_k+2}} + ...$$

with $m'_{k+2} + n'_{k+2} = m_k + n_k + 2 = k+2$. Again we have $a'_{k+2} = 0, 0^*$ e $a_{k+2} = r-1$. Now $a_{k+2} = r-1$ and $k+2$ is an even number. Then $a_{k+3} \in \{0^*, 1^*, ...(r-3)^*\}$ and

$$t' - t = \frac{a'_{k+3}}{(r-2)^{m'_{k+3}} r^{n'_{k+3}}} - \frac{a_{k+3}}{(r-2)^{m_k+1} r^{n_k+2}} + \frac{(r-3)^*}{(r-2)^{m_k+1} r^{n_k+2}} + ... =$$

$$= \frac{a'_{k+3}}{(r-2)^{m'_{k+3}} r^{n'_{k+3}}} + \frac{(r-3)^* - a_{k+3}}{(r-2)^{m_k+1} r^{n_k+2}} + ...$$

with $m'_{k+3} + n'_{k+3} = m_k + n_k + 3 = k+3$. Therefore $a'_{k+3} = 0, 0^*$ e $a_{k+3} = (r-3)^*$. Following this idea we have the result.

Following this idea we can prove the others items.

∎

Corollary 2.6. *Using the notations of lemma 2.5, if $t, t' \in [0,1]$ then $t = t'$ if and only if*

1. $t = a_1...a_{k-1}a_k(r-1)\overline{(r-1)(r-3)^*}$, $t' = a_1...a_{k-1}(a_k+1)\overline{0}$ *or;*
2. $t = a_1...a_{k-1}a_k\overline{(r-1)(r-3)^*}$, $t' = a_1...a_{k-1}(a_k+1)\overline{0}$ *or;*
3. $t = a_1...a_{k-1}a_k(r-3)^*\overline{(r-3)^*(r-1)}$, $t' = a_1...a_{k-1}(a_k+1)\overline{0}$ *or;*
4. $t = a_1...a_{k-1}a_k\overline{(r-3)^*(r-1)}$, $t' = a_1...a_{k-1}(a_k+1)\overline{0}$.

Let $A = \{0, 1, ..., r-1\}$ be a subset of $\mathbb{N}$ and consider

$$\begin{aligned} \psi: \quad & A^{\mathbb{N}} \longrightarrow A^{\mathbb{N}} \\ & (a_i) \longmapsto (b_i) \end{aligned}$$

given by
$b_1 = a_1$;
$b_{2k} = r - 1 - a_{2k}$;
$b_{2k+1} = a_{2k+1}$ if $a_{2k} \in \{0, 0^*, 2n-1, (2n-1)^*, n = 1..., a-2, r-2\}$;
$b_{2k+1} = a_{2k+1} + 2$ if $a_{2k} \in \{2n, (2n)^*, n = 1, ..., a-2, r-1\}$.
Take $x_0 \in R'_{\alpha-1}$ and consider $f : [0,1] \longrightarrow R_{\alpha-1}$ defined as follows:
if $t = \sum_{i=1}^{\infty} a_i (r-2)^{-m_i} r^{-n_i}, (a_i) \in A^{\mathbb{N}}$, then $f(t) = \lim_{n\to\infty} g_{b_1} \circ g_{b_2} \circ ... \circ g_{b_n}(x_0)$ where $\psi(a_1 a_2 ...) = b_1 b_2 ....$

Theorem 2.7. *f is a continue and bijective function satisfying $f(0) = u$ and $f(1) = v$.*

Proof:

1. f is a well defined function.
   We are going to use the following notation: $g_{b_1} ... g_{b_{k-1}} g_{b_k}(z) = b_1 ... b_k$.
   According lemma 2.4 we have
   $u = -1 = 0(r-1)0(r-1)... = \overline{0(r-1)}$.
   $v = -(a-1)\alpha - \alpha^{-1} = (r-1)0(r-1)0... = \overline{(r-1)0}$.
   $w = -1 - \alpha^3 = 2(r-1)0(r-1)0... = 2\overline{(r-1)0}$.
   We have to consider all the cases of corollary 2.6 .

   (a) Let k be an even number and $a_k = 0, 0^*, r-2, (2n-1), (2n-1)^*, n = 1, ..., a-2, a_k+1 = 1, 1^*, r-1, 2n, (2n)^*$.
   Suppose that $t = a_1 ... a_{k-1} a_k (r-1)\overline{(r-1)(r-3)^*} = a_1 ... a_{k-1}(a_k+1)\overline{0} = t'$. Then:
   $f(t) = b_1 ... b_{k-1}(r-1)\overline{(r-1)0}$ and $f(t') = b_1 ... b_{k-1}(r-2)\overline{0(r-1)}$, if $a_k = 0, 0^*$;
   $f(t) = b_1 ... b_{k-1}(r-2n)\overline{(r-1)0}$ and $f(t') = b_1 ... b_{k-1}(r-2n-1)2\overline{(r-1)0}$ if $a_k = 2n-1, (2n-1)^*$;
   $f(t) = b_1 ... b_{k-1}1\overline{(r-1)0}$ and $f(t') = b_1 ... b_{k-1}02\overline{(r-1)0}$ if $a_k = r-2$.
   For all these cases we have $f(t) = f(t')$.

   (b) Let k be an odd number and $a_k = r-2, (2n-1), (2n-1)^*, n = 1, ..., a-2$.

   (c) Let k be an odd number and $a_k = 0,\ 0^*, r-3, (2n), (2n)^*, n = 1, ..., a-3$.

   (d) Let k be an even number and $a_k = r-3, (2n), (2n)^*, n = 1, ..., a-3$.
   For all these cases the proof is similar to (a).

2. Suppose that $f(t) = f(t')$. According to lemma 2.3 we have two possibilities:

   (a) $f(t) = g_{b_1} ... g_{b_{k-1}} g_{b_k}(-1), b_k \in \{1, 3, 5, ..., r-2\}$ and $f(t') = g_{b_1} ... g_{b_{k-1}} g_{b_k+1}(v)$.
   Using the above notations we have
   $f(t) = b_1 ... b_{k-1} b_k \overline{0(r-1)}$ and $f(t') = b_1 ... b_{k-1}(b_k+1)\overline{(r-1)0}$.

      i. k is an even number, $b_k \neq r-2$.
      In this case $b_k = r-1-a_k$ and then $a_k = r-1-b_k$ is an odd number. According to the rules:
      - $b_{k+1} = 0,\ a_k$ odd number $\Rightarrow a_{k+1} = 0$.
      - $b_{k+2} = r-1,\ k+2$ even number $\Rightarrow a_{k+2} = 0$.
      - $b_{k+3} = 0,\ a_{k+2} = 0 \Rightarrow a_{k+3} = 0$.
      Therefore $t = a_1 ... a_{k-1}(r-1-b_k)\overline{0}$.
      We also have $b'_k = b_k + 1 = r-1-a'_k$ and then $a'_k = r-2-b_k \neq 0$ is an even number. According to the rules:
      - $b'_{k+1} = r-1,\ a'_k$ even $\Rightarrow a'_{k+1} = (r-3)^*$.
      - $b'_{k+2} = 0,\ k+2$ even $\Rightarrow a'_{k+2} = r-1$.
      - $b'_{k+3} = r-1,\ a'_{k+2} = r-1 \Rightarrow a_{k+3} = (r-3)^*$.
      Therefore $t' = a_1 ... a_{k-1}(r-2-b_k)\overline{(r-3)^*(r-1)}$ and then $t = t'$.

ii. k is an even number, $b_k = r-2$. In this case $b_k = r-1-a_k$ and then $a_k = r-1-b_k = 1$.

iii. k is an odd number. In this case $b_k = a_k$ or $b_k = a_k + 2$ and then $a_k = b_k$ or $a_k = b_k - 2$. The proof is similar to (i)

(b) $f(t) = g_{b_1}...g_{b_{k-1}}g_{b_k}(-1-\alpha^3), b_k \in \{0, 2, ..., r-3\}$ and $f(t') = g_{b_1}...g_{b_{k-1}}g_{b_k+1}(v)$. Here the proof is similar to (a).

3. f is a continuous function
Let us consider $t, t' \in [0,1]$ given by

$$t = \frac{a_1}{r} + B + \sum_{\substack{k=3 \\ m_k+n_k=k}}^{\infty} \frac{a_k}{(r-2)^{m_k} r^{n_k}}, \; t' = \frac{a'_1}{r} + B' + \sum_{\substack{k=3 \\ m'_k+n'_k=k}}^{\infty} \frac{a'_k}{(r-2)^{m'_k} r^{n'_k}}.$$

If $|t'-t| < (r-2)^m r^n$ then according to lemma (2.5) we have to consider the following cases:
• $t = a_1 \ldots a_{k-1} a_k (r-1)(r-1)(r-3)^*(r-1)(r-3)^* ... a_{N+1} ...$, $t' = a_1 \ldots a_{k-1}(a_k + 1)00...a'_{N+1}...$ if k is even and $a_k = 0,\ 0^\star, r-2, (2n-1), (2n-1)^*, n = 1, ..., a-2$. Using what was done before we have to consider the following:

(a) $f(t) = b_1...b_{k-1}(r-1)(r-1)0(r-1)0...b_{N+1}...$ and $f(t') = b_1...b_{k-1}(r-2)0(r-1)0(r-1)...b'_{N+1}...$, $a_k = 0, 0^*$. Then

$$\begin{aligned} |f(t) - f(t')| &= |g_{b_1} \circ g_{b_2} \circ ... \circ g_{b_{k-1}} \circ g_{r-1}(z_1) - g_{b_1} \circ g_{b_2} \circ ... \circ g_{b_{k-1}} \circ g_{r-2}(z_2)| \leq \\ &\leq |\alpha|^{2(k-1)} |g_{r-1}(z_1) - g_{r-2}(z_2)|. \end{aligned}$$

As $g_{r-2}(u) = g_{r-1}(v)$ then

$$\begin{aligned} |f(t) - f(t')| &\leq |\alpha|^{2(k-1)} \left(|g_{r-1}(z_1) - g_{r-1}(v)| + |g_{r-2}(z_2) - g_{r-2}(u)| \right) \leq \\ &\leq |\alpha|^{2(k-1)} (|\alpha|^2 + |\alpha|^3) diam(R_{\alpha-1}) = |\alpha|^{2k}(1+|\alpha|) diam(R_{\alpha-1}). \end{aligned}$$

(b) $f(t) = b_1...b_{k-1}(r-2n)(r-1)0(r-1)0...b_{N+1}...$ and $f(t') = b_1...b_{k-1}(r-2n-1)2(r-1)0(r-1)0...b'_{N+1}...$, $a_k = 2n-1, (2n-1)^*, n = 1, ..., a-2$.

(c) $f(t) = b_1...b_{k-1}1(r-1)0(r-1)0...b_{N+1}...$ and $f(t') = b_1...b_{k-1}02(r-1)0(r-1)0...b'_{N+1}...$, $a_k = r-2$.

• $t = a_1 \ldots a_{k-1} a_k (r-1)(r-3)^*(r-1)(r-3)^* ... a_{N+1} ...$, $t' = a_1 \ldots a_{k-1}(a_k+1)00...a'_{N+1}...$ if k is odd and $a_k = r-2, (2n-1), (2n-1)^*, n = 1, ..., a-2$.
• $t = a_1 \ldots a_{k-1} a_k (r-3)^*(r-3)^*(r-1)(r-3)^*(r-1) ... a_{N+1} ...$, $t' = a_1 \ldots a_{k-1}(a_k + 1)00...a'_{N+1}...$ if k is odd and $a_k = 0,\ 0^\star, r-3, (2n), (2n)^*, n = 1, ..., a-3$.
• $t = a_1 \ldots a_{k-1} a_k (r-3)^*(r-1)(r-3)^*(r-1) ... a_{N+1} ...$, $t' = a_1 \ldots a_{k-1}(a_k+1)00...a'_{N+1}...$ if k is even and $a_k = r-3, (2n), (2n)^*, n = 1, ..., a-3$.
For these item the proof is similar. In all that cases we conclude that f is a continuous function. ■

Theorem 2.8. *The boundary of $\mathcal{R}_a$, $\forall a > 2$ is homeomorphic to S^1.*

References

[1] S. Akiyama, Cubic Pisot units with finite beta expansions, *Algebraic Number Theory and Diophantine Analysis*, ed. F. Halter-Koch and R.F. Tichy (2000) 11-26.

[2] P. Arnoux, S. Ito, Pisot substitutions and Rauzy fractals, *Bull Belg. Math. Soc. Simon Stevin*, 8 (2001) 181-207.

[3] P. Arnoux, V. Berthé, S. Ito, Discrete planes $\mathbb{Z}^2$-actions, Jacobi-Perron algorithm and substitutions, *Ann. Inst. Fourier*, 52 (2002) 305-349.

[4] J. Bastos, A. Messaoudi, D. Smania, T. Rodrigues, A class of cubic Rauzy fractals, *preprint.*

[5] N. Chekhova, P. Hubert, A. Messaoudi, Propriétés combinatoires, ergodiques et arithmétiques de la substitution de Tribonacci, *J. Theor. Nombres Bordeaux*, 13 (2001) 371-394.

[6] A. Messaoudi, Frontière du fractal de Rauzy et systémes de numération complexe, *Acta Arithmetica*, XCV (2000) 195-224

[7] A. Messaoudi, Propriétés arithmétiques et dynamiques du fractal de Rauzy, *J. Theor. Nombres Bordeaux*, 10 (1998) 135-162.

[8] B. Praggastis, Numerations systems and Markov partitions from self similar tilings, *Trans. Amer. Math. Soc.*, 351 (1999) 3315-3349.

DOI: 10.5540/03.2015.003.01.0013