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Stable bi-maps from closed orientable surfaces to $\mathbb{R} \times \mathbb{R}^2$

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Abstract. In this paper we study stable bi-maps $f = (f_1, f_2) : M \to \mathbb{R} \times \mathbb{R}^2$ from a global viewpoint, where M is a smooth closed orientable surface. We associate a bi-graph to f, so-called $\mathcal{RM}$-graph and study their properties. In this work we are looking for realization conditions for $\mathcal{RM}$-graphs associated to stable bi-maps.

Keywords. Stable maps, $\mathcal{RM}$-graphs, closed surfaces.

1 Introduction

In this work, we use graph theory to study stable maps defined on a smooth closed orientable surface $M \subset \mathbb{R}^3$. Also, we will consider two types of stable maps: $f_1 : M \to \mathbb{R}$ and $f_2 : M \to \mathbb{R}^2$. Stable maps have been investigated by several authors and have many interesting applications (see [2,3,5,6,7,8,10,11,13], for instance).

First of all, let $f_1 : M \to \mathbb{R}$ be a stable map. For this type of map, it is known that the Reeb graph is a global topological invariant associated to f_1 (cf. [4], [12]). The Reeb graph describes the topology of the surface M. Moreover, the Reeb graphs have many applications in Computational Geometry, Computer Graphics, Engineering, Applied Mathematics, etc. We will call the Reeb graph associated to $f_1 : M \to \mathbb{R}$ by $\mathcal{R}$-graph.

Let now $f_2 : M \to \mathbb{R}^2$ be a stable map. For this type of map, by Whitney's Theorem (cf. [13]), the singular set of f_2 (denoted by $\Sigma f_2 \subset M$) consists of curves of double points, possibly containing isolated cusp points. The singular and regular components in the surface M codify relevant information about the stable map f_2. In fact, in [5] graphs with weights on the vertices were introduced as a global topological invariant for stable maps of type $f_2 : M \to \mathbb{R}^2$. We will call the weighted graph associated to $f_2 : M \to \mathbb{R}^2$ by $\mathcal{M}$-graph.

In this work we consider a pair of stable maps (called here stable bi-map) $f = (f_1, f_2) : M \to \mathbb{R} \times \mathbb{R}^2$, then we associated to it a bi-graph $(\mathcal{G}^1, \mathcal{G}^2)$, where $\mathcal{G}^1$ is a $\mathcal{R}$-graph and $\mathcal{G}^2$ is a $\mathcal{M}$-graph. Since the $\mathcal{R}$-graph contributes to determine the position of the maximum and minimum points (local and global) of f_1, and the $\mathcal{M}$-graph contributes to determine the position of the regular regions and singular curves of f_2 in M, we propose the study of some natural questions: Any bi-graph $(\mathcal{G}^1, \mathcal{G}^2)$ can be associated to a stable bi-map $f = (f_1, f_2) : M \to \mathbb{R} \times \mathbb{R}^2$, where M is a smooth closed orientable surface? In other words, every pair of graphs $(\mathcal{G}^1, \mathcal{G}^2)$ is a $\mathcal{RM}$-graph? Otherwise, what conditions should we impose on a pair of graphs $(\mathcal{G}^1, \mathcal{G}^2)$ for it to be a $\mathcal{RM}$-graph?

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2 Stable bi-maps

We begin this section by recall some basic facts.

Definition 2.1. *Two smooth maps $f, g : M \to N$ between two smooth closed orientable manifolds M and N in $\mathbb{R}^n$ are said to be $\mathcal{A}$-equivalent if there are orientation preserving diffeomorphisms, $l : M \to M$ and $k : N \to N$, such that $k \circ f = g \circ l$.*

Definition 2.2. *A smooth map f is said to be* stable *if all maps sufficiently closed to f (in the Whitney C^∞-topology) are $\mathcal{A}$-equivalent to f.*

Definition 2.3. *We say that the pair of smooth maps $f = (f_1, f_2) : M \to \mathbb{R} \times \mathbb{R}^2$ is a stable bi-map if each f_i, $i = 1, 2$, is a stable map.*

Of course that the stability of the pair $f = (f_1, f_2) : M \to \mathbb{R} \times \mathbb{R}^2$, depends on the stability of each f_i, $i = 1, 2$. Remember that:

a) The map $f_1 : M \to \mathbb{R}$ is stable if f_1 is Morse with distinct critical values. That is, if every critical point of f_1 is non-degenerated and each level curve of f_1 has up to one critical point.

b) The map $f_2 : M \to \mathbb{R}^2$ is stable if its singular points are only folds and isolated cups. Remind that a point $p \in M$ is a regular point of f_2 if the map f_2 is a local diffeomorphism around p. Otherwise, the point p is said to be a *singular point*. According to Whitney's Theorem (cf. [13]), the singularities of any stable map $f_2 : M \longrightarrow \mathbb{R}^2$ are (locally) of fold type $(x, y) \mapsto (x, y^2)$ and cusp type $(x, y) \mapsto (x^3 + yx, y)$.

The set of all singular points of f_2, denoted by Σf_2, is called *singular set* of f_2. The singular set of f_2 consists of (finitely many) disjoint embedded closed curves in M. The image of singular the set, $f_2(\Sigma f_2)$, is called the *apparent contour* of f_2. The apparent contour of f_2 is a finite number of immersed closed plane curves with finite number of cups and finite number of transverse intersections and self-intersections (disjoint from the set of cups). The *regular set* of f_2, given by $M \setminus \Sigma f_2$, consists in the set of all regular points of f_2. Since M is a smooth closed orientable surface, the singular set Σf_2 is a finite collection of closed regular simple curves on M made of fold points with possible isolated cusp points that divides M in a set of regular regions.

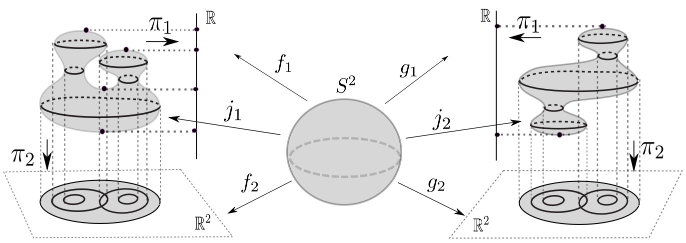

Figura 1: Example of stable bi-maps from sphere.

In this work we are interested in to study stable bi-maps $f = (f_1, f_2) : M \to \mathbb{R} \times \mathbb{R}^2$ where each stable map $f_i : M \to \mathbb{R}^i$, $i = 1, 2$, can be decomposed (locally) as $f_i = \pi_i \circ j$, where $j : M \to \mathbb{R}^3$ is an embedding, $\pi_1 : j(M) \to \mathbb{R}$ and $\pi_2 : j(M) \to \mathbb{R}^2$ are the canonical projections, given by $\pi_1(x, y, z) = z$ and $\pi_2(x, y, z) = (x, y)$, respectively.

The Figure 2 illustrates two different stable bi-maps $f = (f_1, f_2)$ and $g = (g_1, g_2)$ from sphere S^2. The $j_i's$, $i = 1, 2$ indicate two different embedding of M in $\mathbb{R}^3$ and π_i are the canonical projections previously cited, $i = 1, 2$.

2.1 $\mathcal{RM}$-graphs associated to bi-stable maps

Let $j : M \to \mathbb{R}^3$ be an embedding such that the mappings $f_i = \pi_i \circ j$ are stable, where π_i are the canonical projections previously cited, $i = 1, 2$ and M is a smooth closed orientable surface. Then we can consider the stable bi-map $f = (f_1, f_2) : M \to \mathbb{R} \times \mathbb{R}^2$.

Definition 2.4. *Given a stable map $f_1 : M \to \mathbb{R}$ we consider the following equivalence relation on M: $x \sim y \Leftrightarrow f_1(x) = f_1(y)$ and x and y are in the same connected component of $f_1^{-1}(f_1(x))$. The graph given by $M/\sim$ is said to be the* Reeb graph *(or $\mathcal{R}$-graph) associated to $f_1 : M \to \mathbb{R}$ (cf. [1], [2]).*

Definition 2.5. *Given a stable map $f_2 : M \to \mathbb{R}^2$, we define the* Mendes graph *(or $\mathcal{M}$-graph) associated to f_2 (cf. [5], [7]), in the following way:*

1. *The edges and vertices of this weighted graph correspond to the singular curves and the connected components of the regular set, respectively.*
2. *An edge is incident to a vertex if and only if the corresponding singular curve to the edge lies in the boundary of the regular region corresponding to the vertex.*
3. *The weight of a vertex is defined as the genus of the corresponding region.*

Since f_1 is stable we have associated to f_1 its $\mathcal{R}$-graph. Analogously, since f_2 is stable, we have the $\mathcal{M}$-graph associated to f_2. Using these two graphs we define a bi-graph associated to a stable bi-map $f = (f_1, f_2)$ as follows:

Definition 2.6. *If $\mathcal{G}^1$ is the $\mathcal{R}$-graph associated to a stable map $f_1 : M \to \mathbb{R}$ and $\mathcal{G}^2$ is the $\mathcal{M}$-graph associated to a stable map $f_2 : M \to \mathbb{R}^2$, then we say that the pair $(\mathcal{G}^1, \mathcal{G}^2)$ is the* $\mathcal{RM}$-graph *associated to the stable bi-map $f = (f_1, f_2) : M \to \mathbb{R} \times \mathbb{R}^2$.*

The $\mathcal{RM}$-graph will be represented by a bi-graph, as illustrated in the next picture. In each $\mathcal{RM}$-graph the left graph corresponds to the $\mathcal{R}$-graph while the right graph corresponds to the $\mathcal{M}$-graph, respectively. The Figure 2 shows two stable bi-maps from sphere S^2 and their respective $\mathcal{RM}$-graphs. In this picture, notice that the respective apparent contour sets of f_2 and g_2 are the same. This fact suggests that only one of these graphs separately is not able to detect all topological information of M.

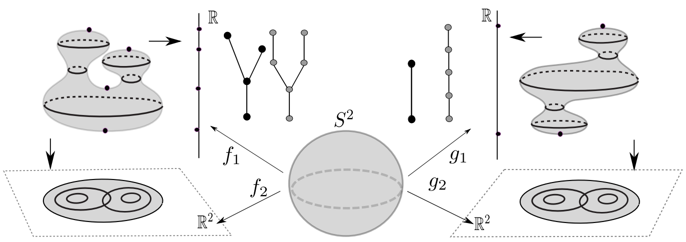

Figura 2: Example of $\mathcal{RM}$-graphs associated to $f = (f_1, f_2)$ and $g = (g_1, g_2)$.

3 Construction of stable bi-maps

In this work we are considering stable bi-maps of type $f = (f_1, f_2) : M \to \mathbb{R} \times \mathbb{R}^2$ which can be decomposed (locally) as $f_i = j \circ \pi_i$, $i = 1, 2$, where j is an embedding from M in $\mathbb{R}^3$ and π_1, π_2

are the canonical projections from $j(M)$ to $\mathbb{R}$ and $\mathbb{R}^2$, respectively. Replacing the embedding j by another embedding from M in $\mathbb{R}^3$, we can obtain new stable bi-maps. This procedure can be done by taking small perturbations of the embedding j, so that they may change or not the images of the projections π_1 and π_2. The new stable bi-maps obtained in this procedure have associated new $\mathcal{RM}$-graphs. Then, it is natural to ask if these changes modify the new $\mathcal{RM}$-graphs or not.

3.1 Elementary Morse transitions

A Morse transition corresponds to an isotopy from a given stable map to another in a different path component of $\mathcal{E}^\infty(M, \mathbb{R})$ (cf. [9]). Thus, a Morse transition allows to transform a stable map $f_1 : M \to \mathbb{R}$ in another $\tilde{f}_1 : M \to \mathbb{R}$ in such a way that their respective $\mathcal{R}$-graphs have a different number of vertices or the same number of vertices with non-compatible labels. A Morse transition T is called *elementary* if the isotopy T transforms f_1 in $\tilde{f}_1$ through one of the following ways:

[**C**] The isotopy T creates a new edge in $\mathcal{R}$-graph of f_1. That is, if $T(0) = f_1$ and its $\mathcal{R}$-graph has s saddles and m max/min points then $T(1) = \tilde{f}_1$ and the $\mathcal{R}$-graph of $\tilde{f}_1$ has $s+1$ saddles and $m+1$ max/min points, with the new saddle and max/min point being connected by a new edge.

[−**C**] It is the inverse transition of **C**. That is, when the isotopy collapses an edge of $\mathcal{R}$-graph of f_1, with the vertices that were removed being previously connected by an edge. In this case, the $\mathcal{R}$-graph of $\tilde{f}_1$ has $s-1$ saddles and $m-1$ max/min points.

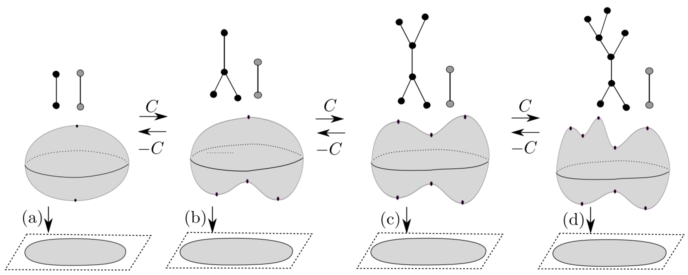

Figura 3: Elementary Morse transitions.

The Figure 3 indicates examples of elementary Morse transitions. Remember that in a $\mathcal{RM}$-graph picture, the left graph corresponds to the $\mathcal{R}$-graph and the right graph is the $\mathcal{M}$-graph. Since elementary Morse transitions do not generate any new critical curve related to projection π_2, the $\mathcal{M}$-graph has no change after **C** or −**C** transitions. In other words, elementary Morse transitions change the $\mathcal{RM}$-graph associated to original stable bi-map $(j \circ \pi_1, j \circ \pi_2)$ changing only its $\mathcal{R}$-graph. Given a $\mathcal{R}$-graph $\mathcal{G}^1$, we say that a **C** transition is a *1-extension* over the graph $\mathcal{G}^1$.

Proposição 3.1. *All pair of trees* $(\mathcal{G}^1(V^1, V^1-1), \mathcal{G}^2(2,1))$ *is a* $\mathcal{RM}$*-graph of some stable bi-map* $f = (f_1, f_2) : S^2 \to \mathbb{R} \times \mathbb{R}^2$*, where* $\mathcal{G}^1(V^1, V^1-1)$ *is a 1-trivalent tree.*

Proof. Since $\mathcal{G}^1(V^1, V^1-1)$ is a 1-trivalent tree, then V^1 is even and $\frac{V^1-2}{2}$ is a integer number. Let $g = (g_1, g_2) : S^2 \to \mathbb{R} \times \mathbb{R}^2$ be the standard stable bi-map, given by $g_i = \pi_i \circ j$, where $j : S^2 \to \mathbb{R}^3$ is an inclusion. Let $(\mathcal{G}^1(2,1), \mathcal{G}^2(2,1))$ be the $\mathcal{RM}$-graph associated to g. After a sequence of $\frac{V^1-2}{2}$ 1-extensions over the $\mathcal{RM}$-graph of g without changing the singular set of g_2, we obtain a new stable bi-map $f = (f_1, f_2) : S^2 \to \mathbb{R} \times \mathbb{R}^2$ which realizes the bi-graph $(\mathcal{G}^1(V^1, V^1-1), \mathcal{G}^2(2,1))$. In fact, each 1-extension increases two edges and two vertices in the $\mathcal{R}$-graph and do not change the $\mathcal{M}$-graph.

3.2 Lips, beaks and swallowtail transitions

In this subsection we will consider transitions that change only the $\mathcal{M}$-graph in a $\mathcal{RM}$-graph associated to a stable bi-map $f = (f_1, f_2) : M \to \mathbb{R} \times \mathbb{R}^2$. They are the same transitions that change the regular and singular sets of f_2, namely the *lips*, denoted by **L**; *beaks transitions*, denoted by **B** and *swallowtail*, denoted by **S**.

We denote by $-$**B**, $-$**L** and $-$**S**, respectively, the inverse transitions of **B**, **L** and **S**.

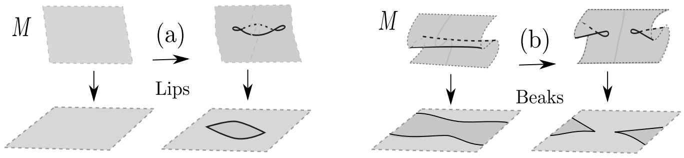

Figura 4: Lips and beaks transitions.

These transitions also change the number of cusps by ± 2 and they are sufficient to show that any tree of zero weight can be realized as a graph of a stable map from S^2 to $\mathbb{R}^2$ (see Theorem 2 in [5]). Let $f_2 : M \to \mathbb{R}^2$ be a stable map and $G^2(V^2, E^2)$ its associated $\mathcal{M}$-graph. Then, the lips transition (indicated by **L**) increases by 1 the number of regions in M (i.e., vertices in V^2) and the number of singular curves in M (i.e., edges in E^2). The swallowtail transition changes the number of cusps but it does not change V^2 and E^2. The beaks transition (indicated by **B**) can be classified in four different cases:

$\mathbf{B}_v^+$: beaks transition increases by 1 the number of regular regions, i.e., it adds 1 vertex and 1 edge on the $\mathcal{M}$-graph;

$\mathbf{B}_v^-$: beaks transition decreases by 1 the number of regular regions, therefore it removes 1 vertex and 1 edge on the $\mathcal{M}$-graph;

$\mathbf{B}_w^+$: beaks transition increases by 1 the weight, maintains the number of regular regions (vertices) but decreases by 1 the number of edges;

$\mathbf{B}_w^-$: beaks transition decreases by 1 the weight, maintains the number of regular regions (vertices) but increases by 1 the number of edges.

The four types of beaks transition are illustrated (locally) in Figure 5, where in the picture X, X_1, Y, Z, Z_1 and Z_2 denote (locally) the regular regions where the transitions hold and the numbers 1 and 2 represent the number of singular curves:

Definition 3.1. *Given a $\mathcal{M}$-graph $\mathcal{G}^2$, we say that a composition of a lips transition with a beaks transition (in this order) is a 2-extension over a $\mathcal{M}$-graph if: (i) a lips transition* ***L*** *creates a singular curve α in M with 2 cusps and 1 regular region; (ii) a beaks transition $-\boldsymbol{B}_v^-$ eliminates the e cusps, dividing α into two new singular curves.*

Lips and beaks transitions can modify the singular set of a stable map from M to the plane, and do not change the singular set of the height function.

We call *line graph*, and denoted it by $\mathcal{L}^2(k)$, a graph with k vertices with degree 2 and $k-1$ edges. Applying 2-extensions we can show that all line graph $\mathcal{L}^2(k)$ is a $\mathcal{M}$-graph of some stable map $f_2 : S^2 \to \mathbb{R}^2$.

Lema 3.2. *All pair of trees $(\mathcal{G}^1(2,1), \mathcal{G}^2(V^2, V^2-1))$ is a $\mathcal{RM}$-graph of some stable bi-map $f = (f_1, f_2) : S^2 \to \mathbb{R} \times \mathbb{R}^2$.*

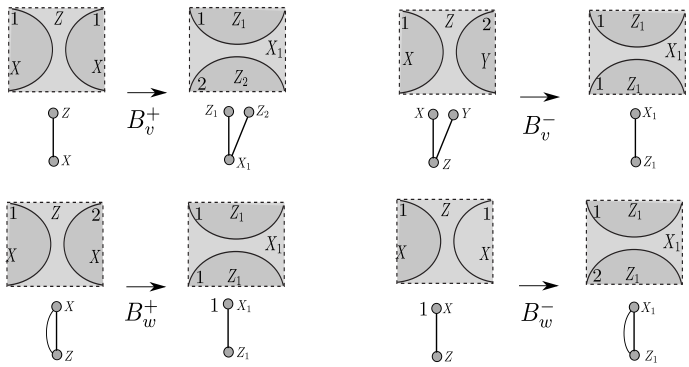

Figura 5: Decomposition of beaks transition.

Proof. Let $g = (g_1, g_2) : S^2 \to \mathbb{R} \times \mathbb{R}^2$ be the pair of canonical maps (given by height function), such that the $\mathcal{RM}$-graph associated to g is $(\mathcal{G}^1(2,1), \mathcal{G}^2(2,1))$, where each g_i is composed by an immersion j from S^2 to $\mathbb{R}^3$ with canonical projections π_i, $i = 1, 2$. Since $\mathcal{G}^2(V^2, V^2 - 1)$ is a tree, let $\mathcal{L}^2(k+1)$ be the biggest line subgraph of $\mathcal{G}^2(V^2, V^2 - 1)$ which connects two peripheral vertices of $\mathcal{G}^2(V^2, V^2 - 1)$, where $k + 1 \leq V^2$. Then, the pair $(\mathcal{G}^1(2,1), \mathcal{L}^2(k+1)$ can be realized as the following:

i) If k is odd, $k - 1$ is even. Passing through a sequence of $\frac{k-1}{2}$ 2-extensions (without changing the singular set of g_1), we obtain a stable bi-map $h = (h_1, h_2) : S^2 \to \mathbb{R} \times \mathbb{R}^2$ which realizes the bi-graph $(\mathcal{G}^1(2,1), \mathcal{L}^2(k+1))$, because each 2-extension increases two edges and two vertices in the $\mathcal{M}$-graph and does not change the $\mathcal{R}$-graph. After this, we can obtain a stable bi-map $f = (f_1, f_2) : S^2 \to \mathbb{R} \times \mathbb{R}^2$, as required, realizing the $\mathcal{RM}$-graph $(\mathcal{G}^1(2,1), \mathcal{G}^2(V^2, V^2 - 1))$, taking $V^2 - k$ lips transitions over $h = (h_1, h_2)$, in convenient regions.

ii) If k is even, we can first obtain a stable bi-map $h = (h_1, h_2)$ which realizes the bi-graph $(\mathcal{G}^1(2,1), \mathcal{L}^2(k+1))$ as done in item i). Then, we can obtain a stable bi-map $f = (f_1, f_2) : S^2 \to \mathbb{R} \times \mathbb{R}^2$, as required, realizing the $\mathcal{RM}$-graph $(\mathcal{G}^1(2,1), \mathcal{G}^2(V^2, V^2 - 1))$, taking $V^2 - k + 1$ lips transitions over $h = (h_1, h_2)$, in convenient regions.

Theorem 3.3. *If $\mathcal{G}^1$ is a 1-trivalent tree and $\mathcal{G}^2$ is a tree with $W = 0$ then the bi-graph $(\mathcal{G}^1, \mathcal{G}^2)$ is a $\mathcal{RM}$-graph of some stable bi-map $f = (f_1, f_2) : S^2 \to \mathbb{R} \times \mathbb{R}^2$.*

Proof. Let $\mathcal{G}^1(V^1, V^1 - 1)$ be a 1-trivalent tree and $\mathcal{G}^2(V^2, V^2 - 1)$ be a tree with $W = 0$. Let $\mathcal{L}^2(k+1)$ be the biggest line subgraph of $\mathcal{G}^2(V^2, V^2 - 1)$. Then by Lemma 3.2, the bi-graph $(\mathcal{G}^1(2,1), \mathcal{L}^2(k+1))$ can be realized by some stable bi-map $g = (g_1, g_2) : S^2 \to \mathbb{R} \times \mathbb{R}^2$. Since V^1 is even and each 1-extension increases 2 vertices and 1 edge to the $\mathcal{R}$-graph, then passing through a sequence of $\frac{V^1-2}{2}$ 1-extension over $g = (g_1, g_2)$ we obtain a stable bi-map $f = (f_1, f_2) : S^2 \to \mathbb{R} \times \mathbb{R}^2$ which realizes the bi-graph $(\mathcal{G}^1(V^1, V^1 - 1), \mathcal{G}^2(V^2, V^2 - 1))$, as required.

Corollary 3.4. *A bi-graph $(\mathcal{G}^1, \mathcal{G}^2)$ is a $\mathcal{RM}$-graph for a stable bi-map $f = (f_1, f_2) : S^2 \to \mathbb{R} \times \mathbb{R}^2$ if and only if $\mathcal{G}^1$ is a tree 1-trivalent and $\mathcal{G}^2$ is a tree with $W = 0$.*

4 Conclusion and Future Work

In this paper we associate a bi-graph, so-called $\mathcal{RM}$-graphs, to stable bi-maps $f = (f_1, f_2) : M \to \mathbb{R} \times \mathbb{R}^2$ from a global viewpoint, where M is a smooth closed orientable surface. Since the

$\mathcal{RM}$-graph captures more information about the topological structure of the surface M than other classic graphs in literature we study the initial properties of the $\mathcal{RM}$-graph looking for information that would not be possible to be read using only one of the graphs separately. As a consequence we present a realization result for a special type of pairs of $\mathcal{RM}$-graphs associated to stable bi-maps (Corollary 3.4). For future work we intend to study a more general realization theorem.

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