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Necessary Optimality Conditions of KKT type for Interval Programming Problems

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Abstract. This work is concerned with mathematical programming problems with inequality constraints in which the objective function is interval-valued. Necessary optimality conditions of Karush-Kuhn-Tucker type are derived through a geometric approach and the use of the generalized Hukuhara differentiability concept.

Keywords. Interval Optimization Problems, Necessary Optimality Conditions, KKT Conditions.

1 Introduction

Considering optimization problems in which the objective function is intervalar-valued is an alternative way to deal with inaccuracies in the data, interval optimization problems have been studied by many researchers and many applications have been examined. See Inuiguchi et al. [1] and Pal et al. [4] and references therein, for instance.

This work is devoted to present optimality conditions for constrained interval optimization problems defined in $\mathbb{R}^n$. Necessary optimality conditions of Karush-Kuhn-Tucker (KKT) type and strict KKT type are developed. As far as we know, strict KKT type optimality conditions are a novelty in the interval context. The KKT and strict KKT type necessary optimality conditions are stated in terms of the generalized Hukuhara (gH) gradient of the interval objective function along with the (classical) gradients of the active constraints. The gH-derivative concept for functions of several variables adopted here was the one given in Stefanini and Arana-Jiménez [5]. This concept is more adequate than others found in the literature, since it really extends the gH-derivative originally defined in Stefanini and Bede [6] for functions on $\mathbb{R}$.

Despite the development of the theory on interval optimization is relatively recent, a great amount of research on this topic has been done so far. For example, optimality conditions for interval optimization problems are presented in Osuna-Gómez et al. [2,3] and Stefanini and Arana-Jiménez [5]. The work [5] brings KKT type optimality conditions for fuzzy optimization problems. As it is well known, such conditions can be particularized for interval problems. By doing this particularization, we see that the KKT type conditions given in [5] are the same as the ones given here, but our approach is different, since we use some geometric ideas. Moreover, we also obtain strict KKT type conditions, which is not done in [5]. To obtain the KKT necessary optimality conditions we use the positive linear independence constraint qualification, which is more general

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than the linear independence constraint qualification assumed in [5]. The characterization of strict KKT solutions is done down the positive linear independence regularity condition. In [2], a different concept of gH-derivative was used, which is not a natural generalization of the case $n = 1$. Necessary and sufficient conditions are given, but for the unconstrained case. The multiobjective constrained case is considered in [3] for one-dimensional problems.

The paper is organized in the following way. In the next section, we set the notation and give some important definitions. In Section 3, we state the interval programming problem this work is concerned with and obtain the necessary optimality conditions. Section 4 is devoted to some concluding words. Finally, we have the references cited throughout the text.

2 Preliminaries

The interval space, denoted by $\mathcal{K}_C(\mathbb{R})$ (or simply $\mathcal{K}_C$), is the set of all convex compact intervals in $\mathbb{R}$, i.e., $\mathcal{K}_C(\mathbb{R}) = \{[\underline{a}, \overline{a}] : \underline{a}, \overline{a} \in \mathbb{R} \text{ and } \underline{a} \leq \overline{a}\}$. We consider the interval space endowed by the Pompeiu-Hausdorff metric, given by $d_H(A, B) = \max\{|\underline{a} - \underline{b}|, |\overline{a} - \overline{b}|\}$ for all $A = [\underline{a}, \overline{a}],\ B = [\underline{b}, \overline{b}] \in \mathcal{K}_C(\mathbb{R})$. It is well known that $(\mathcal{K}_C(\mathbb{R}), d_H)$ is a complete and separable metric space.

Given $A = [\underline{a}, \overline{a}]$, $B = [\underline{b}, \overline{b}] \in \mathcal{K}_C$ and $\lambda \in \mathbb{R}$, the interval arithmetic operations herein used are defined as follows:

$$A + B = [\underline{a} + \underline{b}, \overline{a} + \overline{b}], \text{ and } \lambda \cdot A = \begin{cases} [\lambda\underline{a}, \lambda\overline{a}], & \text{if } \lambda \geq 0, \\ [\lambda\overline{a}, \lambda\underline{a}], & \text{if } \lambda < 0. \end{cases}$$

The gH-difference of two intervals $A, B \in \mathcal{K}_C$ is defined by

$$A \ominus_{gH} B = C \Leftrightarrow \begin{cases} A = B + C, & \text{or} \\ B = A + (-1)C. \end{cases}$$

The cartesian product of n-factors $\mathcal{K}_C$ is denoted by $\mathcal{K}_C^n$, that is, $\mathcal{K}_C^n = \mathcal{K}_C \times \cdots \times \mathcal{K}_C$. Let $A = (A_1, \ldots, A_n), B = (B_1, \ldots, B_n) \in \mathcal{K}_C^n$ and $\lambda \in \mathbb{R}$. The sum and the product by scalar in $\mathcal{K}_C^n$ are defined as follows:

$$\begin{aligned} A \oplus B &= (A_1, \ldots, A_n) \oplus (B_1, \ldots, B_n) = (A_1 + B_1, \ldots, A_n + B_n), \\ \lambda \odot A &= \lambda \odot (A_1, \ldots, A_n) = (\lambda \cdot A_1, \ldots, \lambda \cdot A_n). \end{aligned}$$

By regarding a real number as a degenerated interval, given $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$, we write

$$A \oplus v = ([\underline{a}_1, \overline{a}_1] + [v_1, v_1], \ldots, [\underline{a}_n, \overline{a}_n] + [v_n, v_n]),$$

where $A_i = \left[\underline{a_i}, \overline{a_i}\right] \in \mathcal{K}_C$, $i = 1, \ldots, n$. By $v \in A$ we mean $v_i \in A_i$, $i = 1, \ldots, n$. By the product topology in $\mathbb{R}^n$, we have $\text{int}(A) = (\text{int}(A_1), \ldots, \text{int}(A_n))$, where $\text{int}([\underline{a}_i, \overline{a}_i]) = (\underline{a}_i, \overline{a}_i)$, $i = 1, \ldots, n$.

As mentioned in the introduction, we make use of the gH-differentiability concept, which was recently introduced in Stefanini and Arana-Jiménez [5]. We refer the reader to [5] for the definitions of the gH-derivative, gH-gradient, the gH-directional derivative and the gH-partial derivatives and their properties. Given $F : S \subseteq \mathbb{R}^n \to \mathcal{K}_C$, $x_0 \in S$ and $d \in \mathbb{R}^n$, these are denoted, respectively, as $D_{gH}F(x_0)$, $\nabla_{gH}F(x_0)$, $F'_{gH}(x_0; d)$ and $\frac{\partial_{gH} F}{\partial x_i}(x_0)$, $i = 1, \ldots, n$.

Given $f : \mathbb{R}^n \to \mathbb{R}$ and $x_0 \in \mathbb{R}^n$, the following notation will be used next.

$$(\nabla f)_-(x_0) = \left(\frac{\partial f}{\partial x_1^-}(x_0), \ldots, \frac{\partial f}{\partial x_n^-}(x_0)\right), \quad (\nabla f)_+(x_0) = \left(\frac{\partial f}{\partial x_1^+}(x_0), \ldots, \frac{\partial f}{\partial x_n^+}(x_0)\right).$$

The usual inner product defined on $\mathbb{R}^n$ is denoted by $\langle u, v \rangle$, for all $u, v \in \mathbb{R}^n$. Given $\underline{d},\ \overline{d} \in \mathbb{R}$, we denote $[\underline{d} \vee \overline{d}] := [\min\{\underline{d}, \overline{d}\}, \max\{\underline{d}, \overline{d}\}]$.

By making use of Proposition 11 and Theorem 5 in [5], we obtain the theorem below.

Theorem 2.1. *Let $F : S \to \mathcal{K}_C$ such that $F(x) = \left[\underline{f}(x), \overline{f}(x)\right]$, $x \in S$, where $S \subseteq \mathbb{R}^n$ is open. If F is gH-differentiable at $x_0 \in S$, then for all $d \in \mathbb{R}^n$, one of the following cases holds:*

(i) $\left(\nabla \underline{f}\right)(x_0)$ *and* $\left(\nabla \overline{f}\right)(x_0)$ *both exist and* $F'_{gH}(x_0; d) = \left[\left\langle \left(\nabla \underline{f}\right)(x_0), d\right\rangle \vee \left\langle \left(\nabla \overline{f}\right)(x_0), d\right\rangle\right]$. *Particularly,* $\frac{\partial_{gH} F}{\partial x_i}(x_0) = \left[\frac{\partial \underline{f}}{\partial x_i}(x_0) \vee \frac{\partial \overline{f}}{\partial x_i}(x_0)\right]$, $i = 1, \ldots, n$.

(ii) $\left(\nabla \underline{f}\right)_-(x_0)$, $\left(\nabla \overline{f}\right)_-(x_0)$, $\left(\nabla \underline{f}\right)_+(x_0)$ *and* $\left(\nabla \overline{f}\right)_+(x_0)$ *exist, and satisfy*

$$\left\langle \left(\nabla \underline{f}\right)_-(x_0), d\right\rangle = \left\langle \left(\nabla \overline{f}\right)_+(x_0), d\right\rangle, \ \left\langle \left(\nabla \overline{f}\right)_-(x_0), d\right\rangle = \left\langle (\nabla \underline{f})_+(x_0), d\right\rangle,$$

$$F'_{gH}(x_0; d) = \left[\left\langle \left(\nabla \underline{f}\right)_-(x_0), d\right\rangle \vee \left\langle \left(\nabla \overline{f}\right)_-(x_0), d\right\rangle\right] = \left[\left\langle \left(\nabla \underline{f}\right)_+(x_0), d\right\rangle \vee \left\langle \left(\nabla \overline{f}\right)_+(x_0), d\right\rangle\right].$$

Particularly, $\frac{\partial_{gH} F}{\partial x_i}(x_0) = \left[\frac{\partial \underline{f}}{\partial x_i^-}(x_0) \vee \frac{\partial \overline{f}}{\partial x_i^-}(x_0)\right] = \left[\frac{\partial \underline{f}}{\partial x_i^+}(x_0) \vee \frac{\partial \overline{f}}{\partial x_i^+}(x_0)\right]$, $i = 1, \ldots, n$.

Let $A = [\underline{a}, \overline{a}]$ and $B = [\underline{b}, \overline{b}] \in \mathcal{K}_C$. Herein, the following partial order relations are used:

1. $A \leq_{LU} B$ if and only if either $\underline{a} < \underline{b}$ and $\overline{a} \leq \overline{b}$ or $\underline{a} \leq \underline{b}$ and $\overline{a} < \overline{b}$.
2. $A <_{LU} B$ if and only if $\underline{a} < \underline{b}$ and $\overline{a} < \overline{b}$.

Proposition 2.1 (Stefanini and Arana-Jiménez [5])**.** *Let $A = [\underline{a}, \overline{a}]$, $B = [\underline{b}, \overline{b}]$, $C = [\underline{c}, \overline{c}] \in \mathcal{K}_C$, and $\ll_{LU} \in \{\leq_{LU}, <_{LU}\}$. Then, $A \ll_{LU} B$ if and only if $A \ominus_{gH} B \ll_{LU} [0, 0]$.*

3 Necessary Optimality Conditions

The interval programming problem considered in this work is stated as

$$\begin{array}{ll} \text{minimize} & F(x) = [\underline{f}(x), \overline{f}(x)] \\ \text{subject to} & g_j(x) \leq 0, \ j = 1, \ldots, m, \\ & x \in S \subseteq \mathbb{R}^n, \end{array} \tag{IP}$$

where $\underline{f}, \overline{f} : S \to \mathbb{R}$, $\underline{f}(x) \leq \overline{f}(x)$, $x \in S$, $g_j : S \to \mathbb{R}$, $j = 1, \ldots, m$, and S is open.

The set of all feasible points is denoted by $\mathcal{X}$, that is,

$$\mathcal{X} = \{x \in S : g_j(x) \leq 0, \ j = 1, \ldots, n\}.$$

For each feasible point $x \in \mathcal{X}$, the set of indices of active constraints at x is defined by

$$I(x) = \{j \in \{1, \ldots, n\} : g_j(x) = 0\}.$$

Given $\varepsilon > 0$, $N_\varepsilon(x^*)$ denotes the ε-neighborhood of $x^* \in \mathbb{R}^n$, that is, $N_\varepsilon(x^*) = \{x \in \mathbb{R}^n : |x - x^*| < \varepsilon\}$. Let $x^* \in \mathcal{X}$. Then,

1. x^* is said to be a local LU-solution of (IP) if there exists $\varepsilon > 0$ such that there does not exist $x \in \mathcal{X} \cap N_\varepsilon(x^*)$ with $F(x) \leq_{LU} F(x^*)$.
2. x^* is said to be a local weak LU-solution of (IP) if there exists $\varepsilon > 0$ such that there does not exist $x \in \mathcal{X} \cap N_\varepsilon(x^*)$ with $F(x) <_{LU} F(x^*)$.

It is straightforward to verify that every LU-solution is also a weak LU-solution. So, though all results will be stated for weak LU-solutions, they can be applied to LU-solutions.

We assume throughout this paper that F is gH-differentiable in S and that g_j, $j = 1, \ldots, m$, are continuously differentiable in S.

The result below is a geometrical characterization of weak LU-solutions.

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Proposition 3.1. *If $x^* \in \mathcal{X}$ is a local weak LU-solution of (IP), then*

$$\begin{cases} F'_{gH}(x^*; d) <_{LU} [0, 0], \\ \langle \nabla g_j(x^*), d\rangle < 0, \ j \in I(x^*), \end{cases} \tag{1}$$

does not have any solution $d \in \mathbb{R}^n$.

Proof. We proceed by contradiction, by assuming that system (1) has a solution, say $\hat{d} \in \mathbb{R}^n$. From

$$\lim_{\alpha \to 0^+} \frac{g_j(x^* + \alpha\hat{d}) - g_j(x^*)}{\alpha} = \left\langle \nabla g_j(x^*), \hat{d} \right\rangle < 0, \ j \in I(x^*),$$

it follows that $g_j(x^* + \alpha\hat{d}) = g_j(x^* + \alpha\hat{d}) - g_j(x^*) < 0, \ j \in I(x^*)$, for all $\alpha > 0$ small enough. For j not in $I(x^*)$, we have $g_j(x^*) < 0$, so that, by continuity, $g_j(x^* + \alpha\hat{d}) < 0$, for all $\alpha > 0$ small enough. Then, there exists $\hat{\alpha} > 0$ such that

$$g_j(x^* + \alpha\hat{d}) < 0, \ j = 1, \ldots, m, \ \alpha \in (0, \hat{\alpha}). \tag{2}$$

Moreover, $\hat{d}$ also satisfies

$$F'_{gH}(x^*; \hat{d}) = \lim_{\alpha \to 0^+} \frac{1}{\alpha} \cdot \Big(F(x^* + \alpha\hat{d}) \ominus_{gH} F(x^*) \Big) <_{LU} [0, 0].$$

Let $F'_{gH}(x^*; \hat{d}) = L = [\underline{l}, \overline{l}] \in \mathcal{K}_C$. Let $0 < \varepsilon < -\overline{l}$. By definition of limit, there exists $\tilde{\alpha} > 0$ such that

$$d_H \left(\frac{1}{\alpha} \cdot \Big(F(x^* + \alpha\hat{d}) \ominus_{gH} F(x^*) \Big), L \right) < \varepsilon, \ \alpha \in (0, \tilde{\alpha}).$$

By setting $\frac{1}{\alpha} \cdot \Big(F(x^* + \alpha\hat{d}) \ominus_{gH} F(x^*) \Big) := K_\alpha = \left[\underline{k}_\alpha, \overline{k}_\alpha \right]$, it follows that $\max\{|\underline{k}_\alpha - \underline{l}|, |\overline{k}_\alpha - \overline{l}|\} = d_H(K_\alpha, L) < \varepsilon$, from where $\underline{k}_\alpha < \underline{l} + \varepsilon < \underline{l} - \overline{l} < 0$ and $\overline{k}_\alpha < \overline{l} + \varepsilon < \overline{l} - \overline{l} = 0$. Thus, $K_\alpha <_{LU} [0, 0]$ for $\alpha \in (0, \tilde{\alpha})$ and, from Proposition 2.1, we have

$$F(x^* + \alpha\hat{d}) <_{LU} F(x^*), \ \alpha \in (0, \tilde{\alpha}). \tag{3}$$

Taking $\delta := \min\{\hat{\alpha}, \tilde{\alpha}\}$, it follows from (2) and (3) that $x^* + \alpha\hat{d} \in \mathcal{X}$ along with $F(x^* + \alpha\hat{d}) <_{LU} F(x^*)$ for all $\alpha \in (0, \delta)$. This contradicts the local optimality of x^*. □

Let $x^* \in \mathbb{R}^n$. In what follows, we denote

$$\nabla_\sharp \underline{f}(x^*) = \begin{cases} \nabla \underline{f}(x^*), \text{ if } \underline{f} \text{ is differentiable at } x^*, \\ (\nabla \underline{f})_-(x^*), \text{ otherwise,} \end{cases}$$

$$\nabla_\sharp \overline{f}(x^*) = \begin{cases} \nabla \overline{f}(x^*), \text{ if } \overline{f} \text{ is differentiable at } x^*, \\ (\nabla \overline{f})_-(x^*), \text{ otherwise.} \end{cases}$$

The result stated next is a direct consequence of Theorem 2.1 and Proposition 3.1.

Proposition 3.2. *If $x^* \in \mathcal{X}$ is a local weak LU-solution of (IP), then*

$$\begin{cases} \left\langle \nabla_\sharp \underline{f}(x^*), d \right\rangle < 0, \\ \left\langle \nabla_\sharp \overline{f}(x^*), d \right\rangle < 0, \\ \langle \nabla g_j(x^*), d\rangle < 0, \ j \in I(x^*), \end{cases}$$

does not have any solution $d \in \mathbb{R}^n$.

The following necessary optimality conditions can be seen as of Fritz John type. Below, $0_{\mathbb{R}^n}$ denotes the null vector in $\mathbb{R}^n$.

Theorem 3.1. *Let $x^* \in \mathcal{X}$ be a local weak LU-solution of (IP). Then, there exist $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2$ and $\mu = (\mu_1, \dots, \mu_m) \in \mathbb{R}^m$, not all zero, such that*

$$\lambda_1 \nabla_\sharp \underline{f}(x^*) + \lambda_2 \nabla_\sharp \overline{f}(x^*) + \sum_{j=1}^{m} \mu_j \nabla g_j(x^*) = 0_{\mathbb{R}^n}, \tag{4}$$

$$\mu_j g_j(x^*) = 0, \ \lambda_1, \lambda_2, \mu_j \geq 0, \ j = 1, \dots, m. \tag{5}$$

Proof. The result follows directly from Proposition 3.2 after applying Gordan's Transposition Theorem and defining $\mu_j = 0$ for j not in $I(x^*)$. □

We, now, turn to the KKT type optimality conditions. It is well known that regularity conditions are required to obtain KKT type optimality conditions. Herein, we use the positive linear independence constraint qualification.

Definition 3.1. *The constrains of (IP) are said to satisfy the positive linear independence constraint qualification (PLICQ) at a feasible point $x^* \in \mathcal{X}$ if there do not exist $\beta_j \geq 0$, $j \in I(x^*)$, not all zero, such that*

$$\sum_{j \in I(x^*)} \beta_j \nabla g_j(x^*) = 0_{\mathbb{R}^n}.$$

Theorem 3.2. *Let $x^* \in \mathcal{X}$ be a weak LU-solution of (IP). Assume that the constraints of (IP) satisfy PLICQ at x^*. Then there exists $\mu = (\mu_1, \dots, \mu_m) \in \mathbb{R}^m$ such that*

$$0_{\mathbb{R}^n} \in \nabla_{gH} F(x^*) \oplus \sum_{j=1}^{m} \mu_j \nabla g_j(x^*), \tag{6}$$

$$\mu_j g_j(x^*) = 0, \ \mu_j \geq 0, \ j = 1, \dots, m. \tag{7}$$

Proof. It follows from Theorem 3.1 and the assumption that PLICQ is satisfied, that there exist multipliers $\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\mu}_j$, $j = 1, \dots, n$, with $(\tilde{\lambda}_1, \tilde{\lambda}_2) \neq (0, 0)$, such that the conditions (4)-(5) hold. Thus, normalizing, if necessary, we obtain new multipliers $\lambda_1, \lambda_2, \mu_j$, $j = 1, \dots, n$, satisfying (4)-(5) with $\lambda_1 + \lambda_2 = 1$. Let us denote

$$\frac{\partial \underline{f}}{\partial x_i^\sharp}(x^*) = \begin{cases} \frac{\partial \underline{f}}{\partial x_i}(x^*), & \text{if } \underline{f} \text{ is differentiable at } x^*, \\ \frac{\partial \underline{f}}{\partial x_i^-}(x^*), & \text{otherwise,} \end{cases}$$

$$\frac{\partial \overline{f}}{\partial x_i^\sharp}(x^*) = \begin{cases} \frac{\partial \overline{f}}{\partial x_i}(x^*), & \text{if } \overline{f} \text{ is differentiable at } x^*, \\ \frac{\partial \overline{f}}{\partial x_i^-}(x^*), & \text{otherwise.} \end{cases}$$

It follows from (4) that

$$\lambda_1 \frac{\partial \underline{f}}{\partial x_i^\sharp}(x^*) + \lambda_2 \frac{\partial \overline{f}}{\partial x_i^\sharp}(x^*) + \sum_{j=1}^{m} \mu_j \frac{\partial g_j}{\partial x_i}(x^*) = 0, \ i = 1, \dots, n.$$

Provided $\lambda_1, \lambda_2 \geq 0$ and $\lambda_1 + \lambda_2 = 1$, we see that

$$-\sum_{j=1}^{m} \mu_j \frac{\partial g_j}{\partial x_i}(x^*) \in \left[\frac{\partial \underline{f}}{\partial x_i^\sharp}(x^*) \vee \frac{\partial \overline{f}}{\partial x_i^\sharp}(x^*) \right], \ i = 1, \dots, n,$$

from where, by making use of Theorem 2.1, we obtain

$$0 \in \frac{\partial_{gH} F}{\partial x_i}(x^*) + \sum_{j=1}^{m} \mu_j \frac{\partial g_j}{\partial x_i}(x^*), \ i = 1, \ldots, n.$$

The result is, now, easily obtained. □

Next, we consider the positive linear independence regularity condition. This regularity condition was given in the literature to deal with vector optimization problems. Here, it is adapted to handle the class of interval programming problems. By employing such a condition, we derive a more accurate KKT type theorem, the strict KKT type conditions.

Definition 3.2. *The positive linear independence regularity condition (PLIRC) is said to be satisfied at $x^* \in \mathcal{X}$ if*

(i) there do not exist $\beta_j \geq 0$, $j \in I(x^*)$, not all zero, such that

$$\sum_{j \in I(x^*)} \beta_j \nabla g_j(x^*) = 0_{\mathbb{R}^n};$$

(ii) there does not exist $\lambda_1 > 0$ and $\mu_j \geq 0$, $j \in I(x^*)$, such that

$$\lambda_1 \nabla_{\sharp} \underline{f}(x^*) + \sum_{j \in I(x^*)} \mu_j \nabla g_j(x^*) = 0_{\mathbb{R}^n}$$

or there does not exist $\lambda_2 > 0$ and $\mu_j \geq 0$, $j \in I(x^*)$, such that

$$\lambda_2 \nabla_{\sharp} \overline{f}(x^*) + \sum_{j \in I(x^*)} \mu_j \nabla g_j(x^*) = 0_{\mathbb{R}^n}.$$

Theorem 3.3. *Let $x^* \in \mathcal{X}$ be a weak LU-solution of (IP). Assume that PLIRC holds at x^* and that $\operatorname{int}\left(\frac{\partial_{gH} F}{\partial x_i}(x^*)\right) \neq \emptyset$, $i = 1, \ldots, n$. Then there exists $\mu = (\mu_1, \ldots, \mu_m) \in \mathbb{R}^m$ such that*

$$0_{\mathbb{R}^n} \in \operatorname{int}\left(\nabla_{gH} F(x^*) \oplus \sum_{j=1}^{m} \mu_j \nabla g_j(x^*)\right), \tag{8}$$

$$\mu_j g_j(x^*) = 0, \ \mu_j \geq 0, \ j = 1, \ldots, n. \tag{9}$$

Proof. The proof is very similar to that of Theorem 3.2, but here, my making use of PLIRC, we can ensure that the multipliers λ_1 and λ_2 are both positive along with $\lambda_1 + \lambda_2 = 1$. Therefore,

$$-\sum_{j=1}^{m} \mu_j \frac{\partial g_j}{\partial x_i}(x^*) \in \operatorname{int}\left(\left[\frac{\partial \underline{f}}{\partial x_i^{\sharp}}(x^*) \vee \frac{\partial \overline{f}}{\partial x_i^{\sharp}}(x^*)\right]\right), \ i = 1, \ldots, n,$$

and the result follows. □

Example 3.1. *The interval optimization problem is given as*

$$\begin{array}{ll} \textit{minimize} & F(x_1, x_2) = [x_1 \vee 2x_1] \\ \textit{subject to} & g_1(x_1, x_2) = x_1^2 - 2x_1 - x_2 \leq 0, \\ & g_2(x_1, x_2) = x_1^2 - 2x_1 + x_2 \leq 0, \\ & g_3(x_2, x_2) = -x_1 + x_2^2 \leq 0, \\ & (x_1, x_2) \in S = \mathbb{R}^2. \end{array}$$

It is easy to see that we have $x_1 \geq 0$ *for all feasible point* (x_1, x_2). *Therefore,* $x^* = (0, 0)$ *is clearly a LU-solution of the optimization problem. Let us note that* $x^* = (0, 0)$ *is a switching point of* F *in* $S = \mathbb{R}^2$. *Moreover,* F *is gH-differentiable at* x^* *while it is not H-differentiable at* x^*. *For this optimization problem we have* $I(x^*) = \{1, 2, 3\}$. *Then, the gradients of the active constraints are not linearly independent at* x^*. *But it is easily verifiable they are positively linearly independent, that is, PLICQ is satisfied at* x^*. *It follows from Theorem 3.2 that* x^* *is a KKT-solution of the problem. In fact, (6)-(7) hold, for instance, with* $\mu_1 = \mu_2 = 1/4$ *and* $\mu_3 = 1/2$.

4 Conclusion

It was obtained first-order necessary optimality conditions of KKT and strict KKT type for mathematical programming problems in which the objective function is interval-valued. The characterization of local optimal solutions was made through a geometric approach, resulting in an algebraic condition after applying the Gordan's Theorem of the alternative. The KKT conditions were established under classical constraint qualifications from the literature. It was used the generalized Hukuhara derivative concept, which is, to the best of our knowledge, one of the most general ones for interval-valued maps.

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