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The 2.5D VTI pseudo-acoustic wave equation

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Abstract. The finite-difference method applied to the full 3D wave equation is a rather time-consuming process. However, in the 2.5D case, we can take advantage of the medium symmetry. By taking the Fourier transform with respect to the out-of-plane direction (the symmetry axis) and then, the 3D problem can be reduced to a repeated 2D problem. The third dimension is taken into account by a sum over the corresponding wave-vector component. A criterion for where to end this theoretically infinite sum derives from the stability conditions of the finite-difference schemes employed. In this way, the computation time of the finite-difference calculations can be considerably reduced. The quality of the modelling results obtained with this 2.5D finite-difference scheme is comparable to that obtained using a standard 3D finite-difference scheme. In this work we apply this idea to the anisotropic pseudo-acoustic wave equation.

Keywords. Wave equation, anisotropy, acoustic approximation.

1 Introduction

The notion of two and one-half dimensional (2.5D) wavefield propagation was introduced in seismic applications by Bleistein [4] who considered seismic modelling and Kirchhoff migration for the scalar wave equation. 2.5D algorithms incorporate three-dimensional (3D) wave propagation in a medium which varies in only two dimensions [5]. By 2.5D, we mean fully three-dimensions wave propagation in a medium in which the velocity varies in only two dimensions [8]. It means 2.5D propagation produces a two-dimensional reflector map with amplitudes that approximate the effect of the out-of-plane spreading of the response to a three-dimensional point source. The method assumes that the subsurface has two-dimensional variation only, with the data line being a dip line of the subsurface [7]. Several applications of 2.5D seismic modelling and migration were proposed [6, 10, 13, 16–18].

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On the other hand, Alkhalifah [1] using an acoustic approximation, i.e., setting vertical shear wave (V_{S0}) equals zero, derived an acoustic dispersion relation for a vertical transversely isotropic (VTI) media [14, 15]. And, after that using this acoustic-approximated dispersion relation he derived a pseudo-acoustic wave equation in VTI media [2].

In this work we derive the 2.5D anisotropic (VTI) pseudo-acoustic wave equation and present a numerical experiment to corroborate our claims. Besides, in the 2.5D formulation, we can take advantage of the medium symmetry by taking the Fourier transform with respect to the out-of-plane direction (the symmetry axis) to simulate a 3D problem as a repeated 2D problem. In this way, the computation time of the finite-difference calculations can be considerably reduced.

2 Method

The anisotropic (VTI) pseudo-acoustic wave equation is derived by using an acoustic approximation ($V_{s0} = 0$, V_{s0} is the vertical S-wave velocity) for the dispersion relation in VTI media [1],

$$p_z^2 = \frac{v_n^2}{v_{p0}^2}\left(\frac{1}{v_n^2} - \frac{p_x^2 + p_y^2}{1 - 2\eta v_n^2\,(p_x^2 + p_y^2)}\right), \tag{1}$$

where v_{p0} is the vertical P-wave velocity of the medium and $v_n = v_{p0}\sqrt{1 + 2\delta}$ is the NMO velocity (see [11]). Moreover, the parameter η is given by $\eta = \dfrac{\epsilon - \delta}{1 + 2\delta}$, where ϵ and δ are Thomsen's parameters [14]. Alkhalifah and Tsvankin [3] demonstrated that a representation in terms of just two anisotropic parameters, v_n and η, is sufficient to represent time-related processing. Then, based on the dispersion relation (1), Alkhalifah [2] derived the pseudo-acoustic wave equation for VTI media, which is given by

$$\begin{aligned} &\frac{\partial^4 F}{\partial t^4} - (1 + 2\eta)v_n^2\left(\frac{\partial^4 F}{\partial x^2 \partial t^2} + \frac{\partial^4 F}{\partial y^2 \partial t^2}\right) + 2\eta v_n^2 v_{p0}^2\left(\frac{\partial^4 F}{\partial x^2 \partial z^2} + \frac{\partial^4 F}{\partial y^2 \partial z^2}\right) \\ &\quad = v_{p0}^2 \frac{\partial^4 F}{\partial z^2 \partial t^2} + f(t)\delta(\vec{x} - \vec{x}_s), \end{aligned} \tag{2}$$

where $f(t)$ is a band-limited source and $\vec{x}_s = (x_s, y_s, z_s)$ is the source location. Note that VTI medium is an elastic medium, by not considering the S-wave propagation we assume that only P-wave propagation occurs, obtaining then "pure acoustic" wave propagations in an elastic medium.

We assume that the velocity field is a function of x and z, and that the source and receivers are located in the symmetry plane ($y = 0$). This is the setup for the so-called 2.5D situation. Then, applying the Fourier transform in the out-of-plane direction (y-coordinate), we get

$$\begin{aligned} &\frac{\partial^4 \tilde{F}}{\partial t^4} - (1 + 2\eta)v_n^2 \frac{\partial^4 \tilde{F}}{\partial x^2 \partial t^2} + 2\eta v_n^2 v_{p0}^2 \frac{\partial^4 \tilde{F}}{\partial x^2 \partial z^2} + k^2(1 + 2\eta)v_n^2 \frac{\partial^2 \tilde{F}}{\partial t^2} - 2k^2 \eta v_n^2 v_{p0}^2 \frac{\partial^2 \tilde{F}}{\partial z^2} \\ &\quad = v_{p0}^2 \frac{\partial^4 \tilde{F}}{\partial z^2 \partial t^2} + f(t)\delta(x - x_s)\delta(z - z_s), \end{aligned} \tag{3}$$

where,

$$\tilde{F} = \tilde{F}(x,k,z,t) = \int_{-\infty}^{\infty} F(x,y,z,t)\, e^{-i\,ky}\, dy,$$

and k is the wave number in y-direction.

The solution of equation (3) is given in k-wavenumber domain. Then, by using the inverse Fourier transform at $y = 0$, we obtain the solution of equation (2) in space-time domain, i.e.,

$$F(x,0,z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{F}(x,k,z,t)\, dk \approx \frac{\Delta k}{\pi}\sum_{j=0}^{J} \tilde{F}(x,k_j,z,t).$$

$F(x,0,z,t)$ is the 2.5D solution of equation (2). Some observations must be made. The summation starts in 0, not in $-J$, because $F(x,k,z,t)$ is an even function of k. Theoretically, in the above summation, $J = \infty$. However, because of numerical reasons, we must use a finite value for J, say $J\Delta k = k_{max}$, with k_{max} being an upper bound for the wavenumber. A methodology to calculate J is proposed in the next section.

Numerically, the implementation of the 2.5D finite-difference algorithm is performed by solving equation (3) for a fixed $k = k_j$, $j = 1,\ldots,J$. Then, all contributions are summed up in order to calculate the inverse Fourier transform. It means that the numerical solution of (3) is calculated by solving a 2D finite-difference algorithm J times.

3 Finite-difference approach

Introducing the auxiliary variable $\tilde{P} = \dfrac{\partial^2 \tilde{F}}{\partial t^2}$, equation (3) becomes

$$\begin{aligned}\frac{\partial^2 \tilde{P}}{\partial t^2} &= (1+2\eta)v_n^2\frac{\partial^2 \tilde{P}}{\partial x^2} - 2\eta v_n^2 v_{p0}^2\frac{\partial^4 \tilde{F}}{\partial x^2\partial z^2} - k^2(1+2\eta)v_n^2\,\tilde{P} + 2k^2\eta v_n^2 v_{p0}^2\frac{\partial^2 \tilde{F}}{\partial z^2} + \\ &\quad v_{p0}^2\frac{\partial^2 \tilde{P}}{\partial z^2} + f(t)\delta(x-x_s)\delta(z-z_s).\end{aligned} \tag{4}$$

A set of indices m, n and l is chosen to establish a finite-difference scheme with uniform grid spacings Δx, Δz and Δt in x, z and t directions, respectively: $x_m = x_{min}+m\Delta x$, $z_n = z_{min} + n\Delta z$ and $t_l = t_{min} + l\Delta t$. Consequently, for a fixed k, we denote $\tilde{F}(x_m,k,z_n,t_l) = \tilde{F}^l_{m,n}$ and $\tilde{P}(x_m,k,z_n,t_l) = \tilde{P}^l_{m,n}$. We propose a finite-difference scheme that was chosen to be fourth-order in space and second-order in time. Approximating the temporal derivatives we obtain

$$\left\{\begin{array}{lcl}\tilde{F}^{l+1}_{m,n} & = & 2\tilde{F}^{l}_{m,n} - \tilde{F}^{l-1}_{m,n} + \Delta t^2\tilde{P}^{l}_{m,n} \\ \\ \tilde{P}^{l+1}_{m,n} & = & 2\tilde{P}^{l}_{m,n} - \tilde{P}^{l-1}_{m,n} + \Delta t^2\left(\dfrac{\partial^2 \tilde{P}}{\partial t^2}\right)^l_{m,n}\end{array}\right. ,$$

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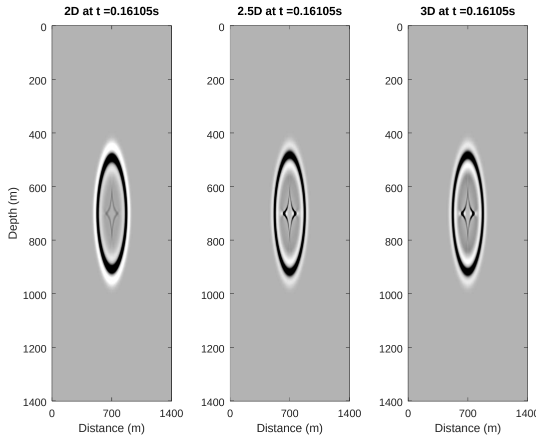
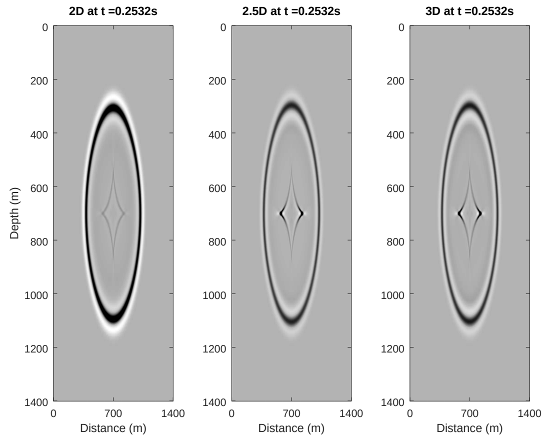
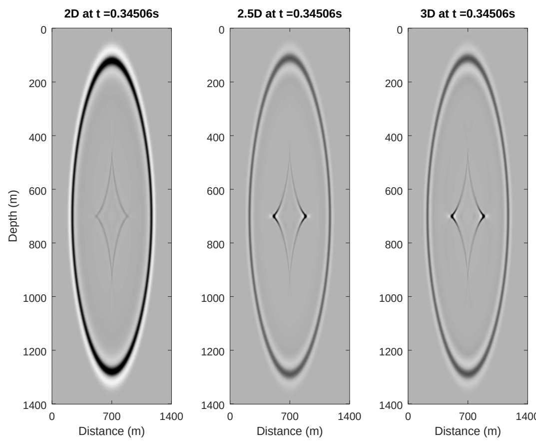


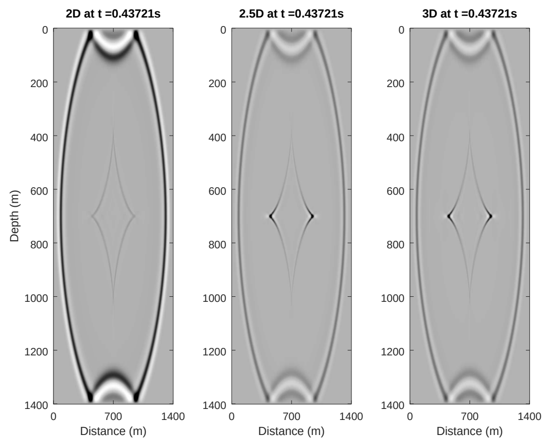
(a) 2D, 2.5D and 3D wavefields at $t = 0.16105$s.



(b) 2D, 2.5D and 3D wavefields at $t = 0.2532$s.



(c) 2D, 2.5D and 3D wavefields at $t = 0.34506$s.



(d) 2D, 2.5D and 3D wavefields at $t = 0.43721$s.

Figure 1: VTI pseudo-acoustic wavefields.

where the second derivative is given by equation (4). The source discretization term is given by

$$f^l_{m,n} = \begin{cases} f(t_l), \quad x_m = x_s \quad \text{and} \quad z_n = z_s, \\ 0, \quad \text{otherwise.} \end{cases}$$

The initial condition is $\tilde{F}^0_{m,n} = 0$ and the boundary conditions are $\tilde{F}^l_{0,n} = \tilde{F}^l_{m,0} = 0$ for all m, n and l.

Alkhalifah [2] demonstrated that the stability condition for the 3D scheme is given by

$$\Delta t < \frac{1}{\sqrt{3}} \min\left(\frac{\Delta x}{\max(v_h)}, \frac{\Delta y}{\max(v_h)}, \frac{\Delta z}{\max(v_v)} \right),$$

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where, v_v and v_h are, respectively, the vertical and horizontal velocities. Assuming $\Delta x = \Delta y = \Delta z = h$ the stability condition can be rewriten as

$$\Delta t < \frac{h}{\sqrt{3}} \min\left(\frac{1}{\max(v_h)}, \frac{1}{\max(v_v)}\right).$$

Also, Alkhalifah states that the finite-difference equations for the VTI pseudo-acoustic wave equation are subjected to the same constraints and rules used in the isotropic case [2]. Therefore, the 2.5D-scheme stability condition is a modification of stability condition derived by Novais and Santos [10], but assuming VTI symmetry. Thus, the stability condition is given by

$$\Delta t < \frac{2h}{\sqrt{k_{max}^2 + 32/(3h^2)}} \min\left(\frac{1}{\max(v_h)}, \frac{1}{\max(v_v)}\right). \tag{5}$$

The constant k_{max} is the maximum out-of-plane wavenumber to be used on the inverse Fourier transform. Following the constraints and rules of the isotropic case, $k_{max} \leqslant \frac{4}{\sqrt{3}h}$. This is an empirical analysis, an accurate and formal determination of the stability conditions must be done. So far, all tests we carried out present the correct (expected) behaviour.

To validate our claims, we FD-propagate a point-source (Ricker wavelet 20Hz) in a 1400m X 1400m grid for 0.46s. The sampling in x- and z-direction is 5m and the time sampling is calculated by (5). The chosen VTI medium is homogeneous and parameterized by $v_{p0} = 1600\, m/s$, $\epsilon = 0.29$ and $\delta = 0.15$.

Figure 1 shows four time-slices, at $t = 0.16105$s, $t = 0.2532$s, $t = 0.34506$s and $t = 0.43721$s. For each slice we present side-by-side the 2D, 2.5D and 3D wavefronts, respectively. As the 2D and 3D geometrical spreading regime are quite different, a scalar was applied to balance the amplitudes and thus allowing a kinematic comparison. Moreover, the 2.5D wavefront looks more symmetrical than the 2D wavefront and similar to the 3D wavefront. This evidences the 2.5D propagation incorporates the 3D characteristics. The artifact in the middle of the wavefront is well known and can be attenuated, therefore it is not a problem for seismic modeling or seismic migration.

Figure 2, on the other hand, presents time-and-space slices. Figure 2(a) presents a comparison between 2.5D and 3D traces along x-direction and z-direction, respectively. The amplitudes and the wavelet shapes coincide, taking into account numerical errors/precision. Figure 2(b) shows a comparison between 2D and 2.5D propagation reinforcing the 2.5D propagation mimics 3D propagation at $y = 0$. A scalar is applied to the 2D traces in order to balance the amplitude to the same order of the 3D (or 2.5D) traces.

The 2.5D formalism is useful in the studies of seismic migration using 2D datasets as input and for seismic modeling [9, 12]. Moreover, the 2.5D FD implementations can be parallelized in a very efficient way because essentially they are a set of J independent 2D wave propagations.

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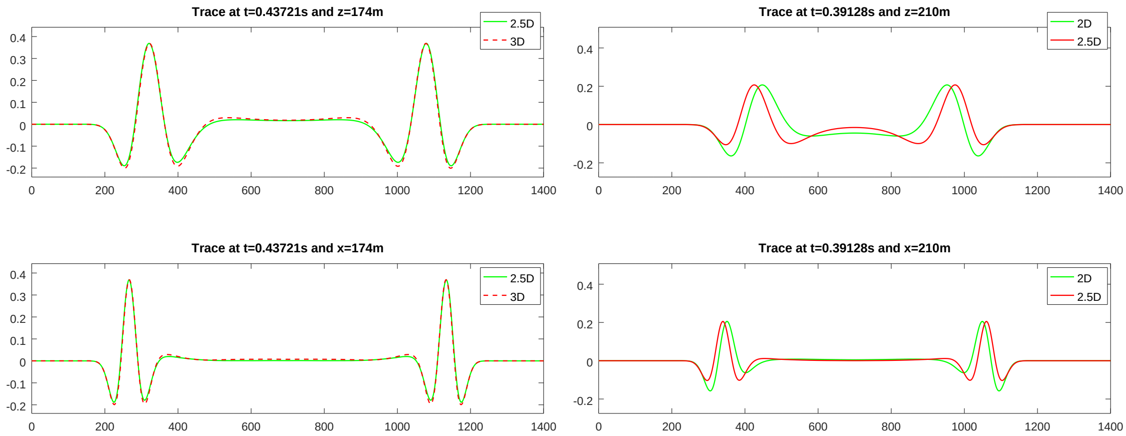

(a) Comparison between 2.5D and 3D. (b) Comparison between 2D and 2.5D.

Figure 2: 2.5D setup validation.

4 Conclusion

In this work, we studied how to implement the 2.5D anisotropic pseudo-acoustic wave equation assuming that the velocity field is a function of x and z, and that the source and receivers are located in the symmetry plane ($y = 0$). Then we observed that the amplitude and phase of the 2.5D wavefield do not change when compared to the 3D propagation. This idea can be of service to simulate 3D propagation using 2D data.

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