Proceeding Series of the Brazilian Society of Applied and Computational Mathematics, Vol. 3, N. 1, 2015.
Trabalho apresentado no XXXV CNMAC, Natal-RN, 2014.

Minimal sets in singularly perturbed systems with three time–scales

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Abstract: *In this work we study three time scale singular perturbation problems*

$$\varepsilon x' = f(\mathbf{x}, \varepsilon, \delta), \qquad y' = g(\mathbf{x}, \varepsilon, \delta), \qquad z' = \delta h(\mathbf{x}, \varepsilon, \delta),$$

where $\mathbf{x} = (x, y, z) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p$, ε *and* δ *are two independent small parameter (*$0 < \varepsilon, \delta \ll 1$*), and* f, g, h *are* C^r *functions, with* $r \geq 1$. *We establish conditions for the existence of compact invariant sets (singular points, periodic and homoclinic orbits) when* $\varepsilon, \delta > 0$. *Our main strategy is to consider three time scales which generate three different limit problems.*

Keywords: *Singular perturbations problems, three time scales*

In this work we study systems with three distinct time–scales. These systems are in general written in the form

$$\varepsilon x' = f(\mathbf{x}, \varepsilon, \delta), \qquad y' = g(\mathbf{x}, \varepsilon, \delta), \qquad z' = \delta h(\mathbf{x}, \varepsilon, \delta), \tag{1}$$

where $\mathbf{x} = (x, y, z) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p$, ε and δ are two independent small parameter ($0 < \varepsilon, \delta \ll 1$), and f, g, h are C^r functions, where r is big enough for our purposes. In system (1) three different time–scales can be derived: a slow time–scale t, an intermediate time–scale $\tau_1 := \frac{t}{\delta}$ and a fast time–scale $\tau_2 := \frac{\tau_1}{\varepsilon}$.

Example. Examples of models involving three time–scales are for instance found in food chain models with a third class of so–called super or top-predators ([7], [1] and [2]) or in hormone secretion models ([5]). For instance, the Rosenzweig–MacArthur model ([8]) for tritrophic food chains (as proposed by [1])

$$\varepsilon x' = x\left(1 - x - \frac{y}{x + b_1}\right), \quad y' = y\left(\frac{x}{x + b_1} - d_1 - \frac{z}{y + b_2}\right), \quad z' = \delta z\left(\frac{y}{y + b_2} - d_2\right), \tag{2}$$

is an example of a problem involving three different time–scales. It is composed of a logistic prey x, a Holling type II predator y and a Holling type II top–predator z. Models with three or more time–scales are also used to study neuronal behavior, in particular to explain firing of neurons or so–called mixed mode oscillations (see [4], [6]).

In this work we develop a mathematical theory in order to study systems (1). Our main goal is to build a theory, inspired by the one given by Fenichel in [3], for systems involving three different time–scales.

1 Statement of the main results

The system (1) is written with respect to the time–scale τ_1 so it is called *intermediate system.* By transforming (1) to the slow and fast variables t and τ_2 we obtain, respectively, the *slow system*

$$\varepsilon\delta x' = f(\mathbf{x},\varepsilon,\delta), \qquad \delta y' = g(\mathbf{x},\varepsilon,\delta), \qquad z' = h(\mathbf{x},\varepsilon,\delta), \tag{3}$$

and the *fast system*

$$x' = f(\mathbf{x},\varepsilon,\delta), \qquad y' = \varepsilon g(\mathbf{x},\varepsilon,\delta), \qquad z' = \varepsilon\delta h(\mathbf{x},\varepsilon,\delta). \tag{4}$$

Remark. To simplify our notation, we will use the notation x' to indicate the derivative with respect to the three time scales. More specifically, for the systems (1), (3) and (4), x' indicates the derivatives $\frac{dx}{d\tau_1}$, $\frac{dx}{dt}$ and $\frac{dx}{d\tau_2}$, respectively.

Note that, for $\varepsilon, \delta \neq 0$, systems (1), (3) and (4) are equivalent. By setting $\varepsilon = \delta = 0$ in (1), (3) and in (4) we obtain three systems with dynamics essentially different: the *intermediate problem*

$$0 = f(\mathbf{x},0,0), \qquad y' = g(\mathbf{x},0,0), \qquad z' = 0, \tag{5}$$

the *reduced problem*

$$0 = f(\mathbf{x},0,0), \qquad 0 = g(\mathbf{x},0,0), \qquad z' = h(\mathbf{x},0,0), \tag{6}$$

and the *layer problem*

$$x' = f(\mathbf{x},0,0), \qquad y' = 0, \qquad z' = 0. \tag{7}$$

For each ε and δ, consider the following sets

$$\mathcal{S}_1^\delta = \{\mathbf{x} \in \mathbb{R}^{n+m+p} : f(\mathbf{x},0,\delta) = 0\}$$

and

$$\mathcal{S}_2^\varepsilon = \{\mathbf{x} \in \mathbb{R}^{n+m+p} : f(\mathbf{x},\varepsilon,0) = g(\mathbf{x},\varepsilon,0) = 0\}.$$

Note that the intermediate and reduced problems (5) and (6) are dynamical systems defined on $\mathcal{S}_1^0$ and $\mathcal{S}_2^0$, respectively. On the other hand $\mathcal{S}_1^0$ is a manifold of singular points for (7). In what follows we refer to $\mathcal{S}_1^0$ and $\mathcal{S}_2^0$ as the *intermediate* and *slow manifolds*, respectively. The reason for these names is that on $\mathcal{S}_1^0$ the intermediate time–scale is dominating and on $\mathcal{S}_2^0$ the slow time–scale predominates.

Following the ideas of the geometric singular perturbation theory [3], our goal will be to prove that one can obtain information on the dynamics of the system (1), for small values of ε and δ, by suitably combining the dynamics of the three limit problems (5), (6) and (7).

Four other systems will also play an important role in our analysis of system (1). By setting $\varepsilon = 0$ in (1) (or in (3)) and in (4) while keeping δ fixed but nonzero, we obtain the *δ–intermediate problem*

$$0 = f(\mathbf{x},0,\delta), \qquad y' = g(\mathbf{x},0,\delta), \qquad z' = \delta h(\mathbf{x},0,\delta), \tag{8}$$

and the *δ–layer problem*

$$x' = f(\mathbf{x},0,\delta), \qquad y' = 0, \qquad z' = 0. \tag{9}$$

DOI: 10.5540/03.2015.003.01.0003

By setting $\delta = 0$ in (1) (or in (4)) and in (3) while keeping ε fixed but nonzero, we obtain the *ε–intermediate problem*

$$\varepsilon x' = f(\mathbf{x}, \varepsilon, 0), \qquad y' = g(\mathbf{x}, \varepsilon, 0), \qquad z' = 0, \tag{10}$$

and the *ε–reduced problem*

$$0 = f(\mathbf{x}, \varepsilon, 0), \qquad 0 = g(\mathbf{x}, \varepsilon, 0), \qquad z' = h(\mathbf{x}, \varepsilon, 0). \tag{11}$$

Note that when both $\varepsilon, \delta \to 0$, the two δ, ε–intermediate problems (8) and (10) become the same limit problem (5). The problems (8) and (11) are dynamical systems defined on the manifolds $\mathcal{S}_1^\delta$ and $\mathcal{S}_2^\varepsilon$, respectively. On the other hand, $\mathcal{S}_1^\delta$ and $\mathcal{S}_2^\varepsilon$ are sets of singular points for the problems (9) and (10), respectively.

Definition 1.1. *We say that system* (1) *is normally hyperbolic at* $\mathbf{x}_0 \in \mathcal{S}_2^0$ *if the real parts of the eigenvalues of the Jacobian matrix*

$$\begin{pmatrix} D_{1,2}\, f(\mathbf{x}_0, 0, 0) \\ D_{1,2}\, g(\mathbf{x}_0, 0, 0) \end{pmatrix}$$

are nonzero. We say that system (1) *is δ–normally hyperbolic at* $\mathbf{x}_0 \in \mathcal{S}_1^\delta$ *if the real parts of the eigenvalues of the Jacobian* $D_1\, f(\mathbf{x}_0, 0, \delta)$ *are nonzero.*

Now we are in position to state our main results.

Theorem A. *Consider the C^r family* (1)*. Let $\mathcal{N} \subseteq \mathcal{S}_2^0$ be a j-dimensional compact normally hyperbolic invariant manifold of the reduced problem* (6) *Then there are $\varepsilon_1 > 0$ and $\delta_1 > 0$ and a C^{r-1} family of manifolds $\{\mathcal{N}_\delta^\varepsilon : \delta \in (0, \delta_1), \varepsilon \in (0, \varepsilon_1)\}$ such that $\mathcal{N}_0^0 = \mathcal{N}$ and $\mathcal{N}_\delta^\varepsilon$ is a hyperbolic invariant manifold of* (1)*.*

Proof. Firstly we use Fenichel's first theorem to study the persistence of $\mathcal{N}$ under δ–perturbations of the system (8). Fenichel's first theorem states that the compact normally hyperbolic invariant manifold $\mathcal{N}$ of the reduced problem (6) persists, for $\delta \neq 0$ small, as an invariant manifold $\mathcal{N}_\delta$ for the system (8). More precisely, there exists $\delta_1 > 0$ and a C^{r-1} family of manifolds $\{\mathcal{N}_\delta : \delta \in (-\delta_1, \delta_1)\}$ such that $\mathcal{N}_0 = \mathcal{N}$ and $\mathcal{N}_\delta$ is a hyperbolic invariant manifold of (8). Now, for each δ fixed, we use again the Fenichel's Theory to study the persistence of $\mathcal{N}_\delta$ under ε–perturbations of the system (1). Note that the system (8) corresponds to the reduced problem associated to the system (1). Fenichel's first theorem says that the compact δ–normally hyperbolic invariant manifold $\mathcal{N}_\delta$ of (8) persists, for $\varepsilon \neq 0$ sufficiently small, for the system (1), that is, there exists $\varepsilon_1 > 0$ and a C^{r-1} family of manifolds $\{\mathcal{N}_\delta^\varepsilon : \varepsilon \in (-\varepsilon_1, \varepsilon_1)\}$ such that $\mathcal{N}_\delta^0 = \mathcal{N}_\delta$ and $\mathcal{N}_\delta^\varepsilon$ is a hyperbolic invariant manifold of (1). This complete the proof of Theorem A. □

Consider system (8) supplemented by the trivial equation $\delta' = 0$

$$0 = f(\mathbf{x}, 0, \delta), \qquad y' = g(\mathbf{x}, 0, \delta), \qquad z' = \delta h(\mathbf{x}, 0, \delta), \qquad \delta' = 0. \tag{12}$$

Let $G(\mathbf{x}, \delta) := (g(\mathbf{x}, 0, \delta), \delta h(\mathbf{x}, 0, \delta), 0)$ be the vector field defined by (12). Assume that the linearization of G at points $(\mathbf{x}, 0)$, such that $\mathbf{x} \in \mathcal{S}_2^0$, has k^s eigenvalues with negative real part and k^u eigenvalues with positive real part. The corresponding stable and unstable eigenspaces have dimensions k^s and k^u, respectively.

Similarly, consider system (4) supplemented by the trivial equation $\varepsilon' = 0$

$$x' = f(\mathbf{x}, \varepsilon, \delta), \qquad y' = \varepsilon g(\mathbf{x}, \varepsilon, \delta), \qquad z' = \varepsilon\delta h(\mathbf{x}, \varepsilon, \delta), \qquad \varepsilon' = 0. \tag{13}$$

Let $H(\mathbf{x}, \varepsilon, \delta) := (f(\mathbf{x}, \varepsilon, \delta), \varepsilon g(\mathbf{x}, \varepsilon, \delta), \varepsilon\delta h(\mathbf{x}, \varepsilon, \delta), 0)$ be the vector field defined by (13). Assume that the linearization of H at points $(\mathbf{x}, 0, \delta)$, such that $\mathbf{x} \in \mathcal{S}_1^\delta$, has l^s and l^u eigenvalues with negative and positive real parts, so that the corresponding stable and unstable eigenspaces have dimensions l^s and l^u, respectively.

Theorem B. *Under the hypotheses of Theorem A, suppose that $\mathcal{N}$ has a $(j+j^s)$–dimensional local stable manifold W^s and a $(j+j^u)$–dimensional local unstable manifold W^u. Then there are $\varepsilon_1 > 0$ and $\delta_1 > 0$ and C^{r-1} families of $(j+j^s+k^s+l^s)$–dimensional and $(j+j^u+k^u+l^u)$–dimensional manifolds $\{\mathcal{W}^s_{\delta,\varepsilon} : \delta \in (0,\delta_1),\ \varepsilon \in (0,\varepsilon_1)\}$ and $\{\mathcal{W}^u_{\delta,\varepsilon} : \delta \in (0,\delta_1),\ \varepsilon \in (0,\varepsilon_1)\}$ such that for $\delta, \varepsilon > 0$ the manifolds $\{\mathcal{W}^s_{\delta,\varepsilon}\}$ and $\{\mathcal{W}^u_{\delta,\varepsilon}\}$ are local stable and unstable manifolds of $\mathcal{N}^\varepsilon_\delta$, respectively.*

Proof. Fenichel's second theorem says that, for small nonzero δ, the invariant manifold $\mathcal{N}_\delta$ of (8) has a $(j+j^s+k^s)$–dimensional local stable manifold $\mathcal{W}^s_\delta$ and a $(j+j^u+k^u)$–dimensional local unstable manifold $\mathcal{W}^u_\delta$. Now, for each δ fixed, Fenichel's second theorem also states that, for $\varepsilon \neq 0$ sufficiently small, the invariant manifold $\mathcal{N}^\varepsilon_\delta$ of (1) has a $(j+j^s+k^s+l^s)$–dimensional local stable manifold $\mathcal{W}^s_{\delta,\varepsilon}$ and a $(j+j^u+k^u+l^u)$–dimensional local unstable manifold $\mathcal{W}^u_\delta$. This complete the proof of Theorem B. □

2 Examples

In this section we give some examples where Theorems A and B are applied.

Example 1. Consider the following 3–dimensional system

$$\varepsilon x' = x - \varepsilon + \delta, \qquad y' = -y + \varepsilon + \delta, \qquad z' = \delta z. \tag{14}$$

The intermediate and slow manifolds $\mathcal{S}^0_1$ and $\mathcal{S}^0_2$ are given, respectively, by $\mathcal{S}^0_1 = \{(x,y,z) \in \mathbb{R}^3 : x = 0\}$ and $\mathcal{S}^0_2 = \{(x,y,z) \in \mathbb{R}^3 : x = y = 0\}$. On $\mathcal{S}^0_1$ we have defined the intermediate problem

$$0 = x, \qquad y' = -y, \qquad z' = 0, \tag{15}$$

and on $\mathcal{S}^0_2$ we have defined the reduced problem

$$0 = x, \qquad 0 = y, \qquad z' = z. \tag{16}$$

Moreover, the layer problem is given by

$$x' = x, \qquad y' = 0, \qquad z' = 0. \tag{17}$$

Figure 1 illustrates the phase portraits of the problems (15), (16) and (17), respectively.

By using the notation given in Theorem B, note that we have $j = 0$, $j^s = 0$, $j^u = 1$, $k^s = 1$, $k^u = 0$, $l^s = 0$ e $l^u = 1$. We can then apply Theorems A and B at the normally hyperbolic singular point $\mathcal{N} = (0,0,0)$ of (16). Applying Theorem A, we obtain for small nonzero δ, ε, a family $\mathcal{N}^\varepsilon_\delta$ of hyperbolic singular points of (14). In fact, the family $\mathcal{N}^\varepsilon_\delta$ of singular points is given by $(\varepsilon - \delta, \varepsilon + \delta, 0)$. Applying Theorem B, we can conclude that each singular point $\mathcal{N}^\varepsilon_\delta$ has a 1–dimensional local stable manifold $\mathcal{W}^s_{\delta,\varepsilon}$ and a 2–dimensional local unstable manifold $\mathcal{W}^u_{\delta,\varepsilon}$.

In the next example we study the dynamics of a biological model.

Example 2. Consider the Rosenzweig–MacArthur model ([8]) for tritrophic food chains (as

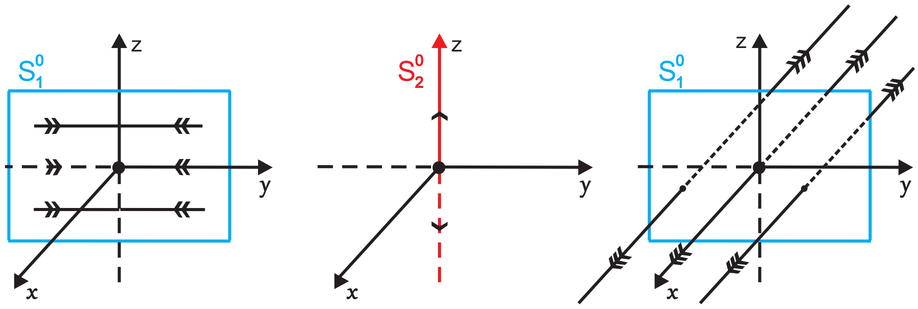

Figura 1: Phase portraits of the systems (15), (16) and (17), respectively.

proposed by [1])

$$\begin{aligned}\varepsilon x' &= x\left(1 - x - \frac{y}{x+b_1}\right) = xf(x,y),\\ y' &= y\left(\frac{x}{x+b_1} - d_1 - \frac{z}{y+b_2}\right) = yg(x,y,z),\\ z' &= \delta z\left(\frac{y}{y+b_2} - d_2\right) = \delta z h(y),\end{aligned} \tag{18}$$

where x, y and z are 1–dimensional variables which represent a logistic prey, a Holling type II predator and a Holling type II top–predator, respectively. All parameters b_1, b_2, d_1 and d_2 are assumed to be positive and less than 1, i.e., $0 < b_1, b_2, d_1, d_2 < 1$. We note that all discussions below are restricted to the first octant, i.e., $x \geq 0$, $y \geq 0$ e $z \geq 0$.

The intermediate and slow manifolds $\mathcal{S}_1^0$ and $\mathcal{S}_2^0$ are given, respectively, by $\mathcal{S}_1^0 = \{xf(x,y) = 0\} = \{(x,y,z) : x = 0\} \cup \{(x,y,z) : y = (1-x)(b_1+x)\} = M_1 \cup M_2$ and $\mathcal{S}_2^0 = \{xf(x,y) = yg(x,y,z) = 0\} = \{(x,y,z) : x = y = 0\} \cup \{(x,y,z) : x = 1, y = 0\} = M_3 \cup M_4$.

The intermediate problem is a dynamical system defined on $\mathcal{S}_1^0 = M_1 \cup M_2$. On M_1 it is given by

$$y' = y\left(-d_1 - \frac{z}{y+b_2}\right), \qquad z' = 0, \tag{19}$$

and on M_2 it becomes

$$y' = y\left(\frac{x}{x+b_1} - d_1 - \frac{z}{y+b_2}\right), \qquad z' = 0. \tag{20}$$

The reduced problem is a dynamical system defined on $\mathcal{S}_2^0 = M_3 \cup M_4$. On both M_3 and M_4 it is given by

$$z' = -d_2 z. \tag{21}$$

The layer problem is given by

$$x' = x\left(1 - x - \frac{y}{x+b_1}\right), \qquad y' = 0, \qquad z' = 0. \tag{22}$$

Figure 2 illustrates the phase portraits of the reduced and layer problems, respectively. Figure 3 illustrates the phase portraits of the systems (19) and (20), respectively. For the phase portrait of (20) we are assuming that $1/(1+b_1) > d_1$.

Note that $\mathcal{N} = (0,0,0)$ and $\mathcal{M} = (1,0,0)$ are singular points of (21). Moreover, according with item (i) of the Definition 1.1, system (18) is normally hyperbolic at $\mathcal{N}$ and $\mathcal{M}$ (for the point $\mathcal{M}$ we are supposing that $d_1 \neq 1/(1+b_1)$). Applying Theorem A, we obtain for small

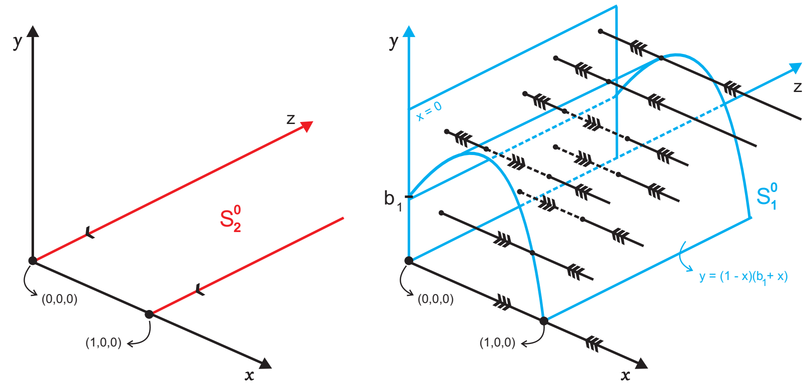

Figura 2: Phase portraits of the systems (21) and (22), respectively.

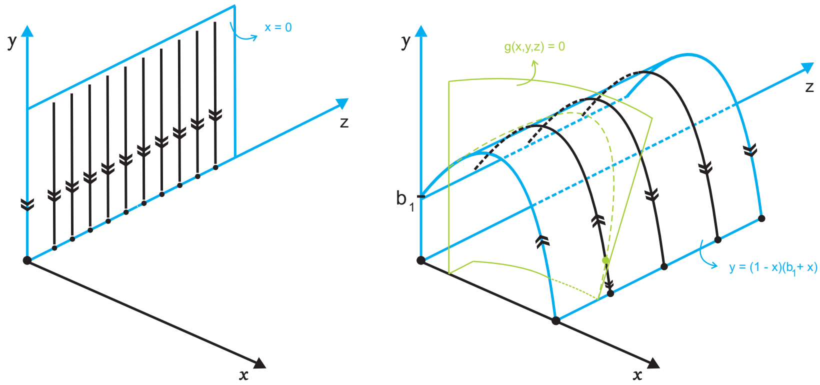

Figura 3: Phase portraits of the systems (19) and (20), respectively.

nonzero δ, ε, families $\mathcal{N}_\delta^\varepsilon$ and $\mathcal{M}_\delta^\varepsilon$ of hyperbolic singular points of (18). In fact, the persistent singular points $\mathcal{N}_\delta^\varepsilon$ and $\mathcal{M}_\delta^\varepsilon$ are given by $(0,0,0)$ and $(1,0,0)$, respectively.

By using the notation given in Theorem B, we have that: for the point $\mathcal{N}$, $j=0$, $j^s=1$, $j^u=0$, $k^s=1$, $k^u=0$, $l^s=0$ e $l^u=1$, and for the point $\mathcal{M}$, $j=0$, $j^s=1$, $j^u=0$, $k^s=0$, $k^u=1$, $l^s=1$ e $l^u=0$. Applying Theorem B, we can conclude that each singular point $\mathcal{N}_\delta^\varepsilon$ has a 2–dimensional local stable manifold $\mathcal{W}_{\delta,\varepsilon}^s$ and a 1–dimensional local unstable manifold $\mathcal{W}_{\delta,\varepsilon}^u$. Moreover, each singular point $\mathcal{M}_\delta^\varepsilon$ has a 2–dimensional local stable manifold $\overline{\mathcal{W}}_{\delta,\varepsilon}^s$ and a 1–dimensional local unstable manifold $\overline{\mathcal{W}}_{\delta,\varepsilon}^u$.

Example 3. Consider the following 4–dimensional system

$$\begin{aligned} \varepsilon x' &= x - z_1 + \delta + \varepsilon = f(x, z_1, \delta, \varepsilon), \\ y' &= -y - z_2 + \delta - \varepsilon = g(y, z_2, \delta, \varepsilon), \\ z_1' &= \delta h_1(x, z_1, z_2), \\ z_2' &= \delta h_2(y, z_1, z_2), \end{aligned} \tag{23}$$

where $h_1(x, z_1, z_2) = -z_2 - z_1(-1 + z_1^2 + z_2^2) + (x - z_1)^2$ and $h_2(y, z_1, z_2) = z_1 - z_2(-1 + z_1^2 + z_2^2) - (y + z_2)^2$. The intermediate and slow manifolds $\mathcal{S}_1^0$ and $\mathcal{S}_2^0$ are given, respectively, by $\mathcal{S}_1^0 = \{(z_1, y, z_1, z_2) \in \mathbb{R}^4 : y, z_1, z_2 \in \mathbb{R}\}$ and $\mathcal{S}_2^0 = \{(z_1, -z_2, z_1, z_2) \in \mathbb{R}^4 : z_1, z_2 \in \mathbb{R}\}$. Note that $\mathcal{S}_1^0$ and $\mathcal{S}_2^0$ are manifolds of dimension 3 and 2, respectively.

On $\mathcal{S}_1^0$ we have defined the intermediate problem

$$x = z_1, \quad y' = -y - z_2, \quad z_1' = 0, \quad z_2' = 0, \tag{24}$$

and on $\mathcal{S}_2^0$ we have defined the reduced problem

$$x = z_1, \quad y = -z_2, \quad z_1' = -z_2 - z_1(-1 + z_1^2 + z_2^2), \quad z_2' = z_1 - z_2(-1 + z_1^2 + z_2^2). \tag{25}$$

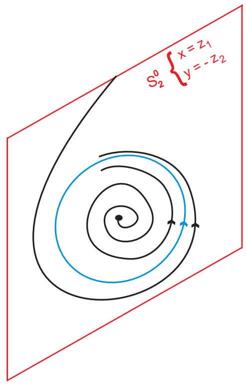

Figura 4: Phase portrait of the system (25).

Moreover, the layer problem is given by

$$x' = x - z_1, \quad y' = 0, \quad z_1' = 0, \quad z_2' = 0. \tag{26}$$

For the phase portrait of the reduced problem we can use polar coordinates $z_1 = r\cos\theta$ and $z_2 = r\sin\theta$. Using these coordinates it is easy to see that the system (25) presents a singular point $\mathcal{P}$ at the origin and a stable limit cycle Γ, as shown Figure 4.

According with item (i) of the Definition 1.1, all points of the slow manifold are normally hyperbolic. Applying Theorem A, we obtain for small nonzero δ, ε, families $\mathcal{P}_\delta^\varepsilon$ and $\Gamma_\delta^\varepsilon$ of hyperbolic singular points and limit cycles of (23), respectively, such that $\mathcal{P}_0^0 = \mathcal{P}$ and $\Gamma_0^0 = \Gamma$. By using the notation given in Theorem B, we have that: for the point $\mathcal{P}$, $j = 0$, $j^s = 0$, $j^u = 2$, $k^s = 1$, $k^u = 0$, $l^s = 0$ e $l^u = 1$, and for the limit cycle Γ, $j = 1$, $j^s = 1$, $j^u = -1$, $k^s = 1$, $k^u = 0$, $l^s = 0$ e $l^u = 1$. In agreement with Theorem B, each singular point $\mathcal{P}_\delta^\varepsilon$ has an 1–dimensional local stable manifold $\mathcal{P}_{\delta,\varepsilon}^s$ and a 3–dimensional local unstable manifold $\mathcal{P}_{\delta,\varepsilon}^u$. Each limit cycle $\Gamma_\delta^\varepsilon$ has a 3–dimensional local stable manifold $\Gamma_{\delta,\varepsilon}^s$ and an 1–dimensional local unstable manifold $\Gamma_{\delta,\varepsilon}^u$.

Referências

[1] B. Deng, Food chain chaos due to junction–fold point, *Chaos*, 11(3) (2001), 514–525.

[2] B. Deng and G. Hines, *Food chain chaos due to Shilnikov's orbit*, Chaos **12(3)** (2002), 533–538.

[3] N. Fenichel, *Geometric singular perturbation theory for ordinary differential equations*, J. Diff. Equations **31** (1979), 53–98.

[4] M. Krupa, N. Popović and N. Kopell *Mixed–mode oscillations in three time–scale systems: a prototypical example*, SIAM Appl. Dyn. Syst. 7(2) (2008), 361–420.

[5] W. Kunpasuruang, Y. Lenbury and G. Hek, *A nonlinear mathmatical model for pulsatile discharges of luteinizing hormone mediated by hypothalamic and extra–hypothalamic pathways*, Math Models Methods Appl. Sci. **12(5)** (2002), 607–624.

[6] G.S. Medvedev and J.E. Cisternas, *Multimodal regimes in a compartmental model of the dopamine neuron*, Physica D **194(3-4)** (2004), 333–356.

[7] S. Muratori and S. Rinaldi, *Low- and high–frequency oscillations in three–dimensional food chain systems*, SIAM J. Appl. Math. **52** (1992), 1688–1706.

[8] M.L. Rosenzweig and R.H. MacArthur, *Graphical representation and stability conditions of predator–prey interactions*, Am. Nat. **97** (1963), 209–223.

