Proceeding Series of the Brazilian Society of Applied and Computational Mathematics, Vol. 3, N. 1, 2015.
Trabalho apresentado no XXXV CNMAC, Natal-RN, 2014.

Singularly Perturbed Discontinuous Vector Fields

Paulo Ricardo da Silva, **Jaime Rezende de Moraes**,
Depto de Matemática, IBILCE, UNESP,
15054-000, São José do Rio Preto, SP
E-mail: prs@ibilce.unesp.br, jrezmoraes@gmail.com,

Pedro Toniol Cardin
Departamento de Matemática, FEIS, UNESP
15385-000, Ilha Solteira, SP
E-mail: pedrocardin@mat.feis.unesp.br.

Keywords: Filippov systems, singular perturbation, tangency points

Abstract. In this article we deal with singularly perturbed vector fields Z_ε expressed by

$$\dot{x} = \begin{cases} F(x,y,\varepsilon) & \text{if} \quad h(x,y,\varepsilon) \leq 0, \\ G(x,y,\varepsilon) & \text{if} \quad h(x,y,\varepsilon) \geq 0, \end{cases} \qquad \varepsilon \dot{y} = H(x,y,\varepsilon), \tag{1}$$

where $\varepsilon \in \mathbb{R}$ is a small parameter, $x \in \mathbb{R}^n, n \geq 2$, and $y \in \mathbb{R}$ denote the slow and fast variables, respectively, and F, G, h and H are smooth maps. We study the effect of singular perturbations at typical singularities of Z_0. Special attention will be dedicated to those points satisfying $q \in \{h(x,y,0) = 0\} \cap \{H(x,y,0) = 0\}$ where F or G is tangent to $\{h(x,y,0) = 0\}$. The persistence and the stability properties of those objects are investigated.

1 Introduction

Let $U \subset \mathbb{R}^n$ be an open set. We denote by $C^r(U,\mathbb{R}^n)$ the set of all vector fields of class C^r defined on U, with $r \geq 1$, endowed with the C^r–topology. The simplest case of a Filippov system is when the phase space is composed by two domains such that for each domain a different ODE (ordinary differential equation) governs the dynamics, namely

$$\dot{x} = Z(x) = \begin{cases} F(x) & \text{if} \quad h(x) \leq 0, \\ G(x) & \text{if} \quad h(x) \geq 0. \end{cases} \tag{2}$$

In equation (2), $F, G \in C^r(U,\mathbb{R}^n)$ and $h : U \to \mathbb{R}$ is a smooth function having $0 \in \mathbb{R}$ as a regular value. The common boundary $\mathcal{M} = \{x \in U \mid h(x) = 0\}$ between the domains $\mathcal{M}_- = \{x \in U \mid h(x) \leq 0\}$ and $\mathcal{M}_+ = \{x \in U \mid h(x) \geq 0\}$ is called *switching manifold*. We will use the notation $Z = (F,G)$ to represent the Filippov system (2) and denote by $\Omega^r(U)$ the set of all vector fields Z of the form (2) defined on U.

We also use $Fh(p) = F(p) \cdot \nabla h(p)$ for the scalar product in $\mathbb{R}^n$ between the vector field $F : U \to \mathbb{R}^n$ and the gradient of the function $h : U \to \mathbb{R}$.

On the switching manifold $\mathcal{M}$ the following open sets are distinguished:

- *Sewing region*: $\mathcal{M}^1 = \{p \in \mathcal{M} : [Fh(p)][Gh(p)] > 0\}$;
- *Escaping region*: $\mathcal{M}^2 = \{p \in \mathcal{M} : Fh(p) < 0, Gh(p) > 0\}$;

- *Sliding region*: $\mathcal{M}^3 = \{p \in \mathcal{M} : Fh(p) > 0, Gh(p) < 0\}$.

The definitions of these three regions exclude the so-called tangency points, that is, points where one of the two vector fields F or G is tangent to $\mathcal{M}$. They are characterized by $p \in \mathcal{M}$ such that $Fh(p) = 0$ or $Gh(p) = 0$. Generically speaking, these points are on the boundary of the regions $\mathcal{M}^1$, $\mathcal{M}^2$ and $\mathcal{M}^3$, which we denote by $\partial\mathcal{M}^1$, $\partial\mathcal{M}^2$ and $\partial\mathcal{M}^3$, respectively. Tangency points include the case $F(p) = 0$ or $G(p) = 0$, that is, when one of the two vector fields has an equilibrium point at $\mathcal{M}$. We define two types of tangency between a smooth vector field and a manifold, which will be used in paper. We say that a smooth vector field F has a *fold* or quadratic tangency with $\mathcal{M} = \{h(x) = 0\}$ at a point $p \in \mathcal{M}$ provided $Fh(p) = 0$ and $F^2h(p) \neq 0$. A smooth vector field F has a *cusp* or cubic tangency with $\mathcal{M} = \{h(x) = 0\}$ at a point $p \in \mathcal{M}$ provided $Fh(p) = F^2h(p) = 0$, $F^3h(p) \neq 0$, and the set $\{\nabla h(p), \nabla(Fh)(p), \nabla(F^2h)(p)\}$ is linearly independent.

If a point of the phase space which is moving on an orbit of $Z = (F, G)$ falls onto $\mathcal{M}^1$ then it crosses $\mathcal{M}^1$ over to another part of the space. In $\mathcal{M}^2$ and $\mathcal{M}^3$, the definition of the local orbit is given by the Filippov convention [3]. We consider the vector field f_s which is the linear convex combination of F and G tangent to $\mathcal{M}$, that is

$$\dot{x} = f_s(x) = \frac{[\nabla h(x)F(x)]G(x) - [\nabla h(x)G(x)]F(x)}{\nabla h(x)[F(x) - G(x)]}. \tag{3}$$

We call f_s the *sliding vector field* associated to the Filippov system (2), independently whether it is defined in the sliding or escaping region. Solutions of $Z = (F, G)$ through points of $\mathcal{M}^2 \cup \mathcal{M}^3$ follow the orbit of f_s. The singularities of a Filippov vector field (2) are

- $p \in \mathcal{M}_\pm$ such that p is an equilibrium of F or G, that is, $F(p) = 0$ or $G(p) = 0$, respectively;
- $p \in \mathcal{M}^2 \cup \mathcal{M}^3$ such that p is an equilibrium of f_s, that is, $f_s(p) = 0$;
- $p \in \partial\mathcal{M}^1 \cup \partial\mathcal{M}^2 \cup \partial\mathcal{M}^3$, that is, the tangency points ($Fh(p) = 0$ or $Gh(p) = 0$).

Fenichel (see [2, 4]) proved that, in smooth dynamical systems, any structure which persists under regular perturbation also persists under singular perturbation. In [1] we extended this theory for the sliding vector field associated to the reduced problem of (11), that is, we study how sliding mode in Filippov systems is affected by singular perturbations. Now we analyze the effect of singular perturbations at the tangency points.

For each $\varepsilon \geq 0$ we will denote by $\mathcal{M}_\varepsilon$ and $\mathcal{S}_\varepsilon$ the sets $\mathcal{M}_\varepsilon = \{h(x, y, \varepsilon) = 0\}$ and $\mathcal{S}_\varepsilon = \{H(x, y, \varepsilon) = 0\}$. Note that $\mathcal{M}_\varepsilon$ is the switching manifold. The set $\mathcal{S}_0$ is called the *slow manifold* of the singular perturbation problem (1). Here we are supposing that $\mathcal{M}_0$ and $\mathcal{S}_0$ are in general position, i.e., $\nabla h(p)$ and $\nabla H(p)$ are linearly independent for any $p \in \mathcal{M}_0 \cap \mathcal{S}_0$. Throughout this article we will assume that the equation $H(x, y, \varepsilon) = 0$ can be solved by $y = f_\varepsilon(x)$, for all $\varepsilon \geq 0$.

For $\varepsilon = 0$ in (1) we have the so–called *reduced problem*

$$\dot{x} = \begin{cases} \tilde{F}(x) & \text{if} \quad \tilde{h}(x) \leq 0, \\ \tilde{G}(x) & \text{if} \quad \tilde{h}(x) \geq 0, \end{cases} \qquad 0 = H(x, y, 0), \tag{4}$$

where $\tilde{F}(x) = F(x, f_0(x), 0)$, $\tilde{G}(x) = G(x, f_0(x), 0)$ and $\tilde{h}(x) = h(x, f_0(x), 0)$. The reduced problem (9) is a dynamical system defined on the manifold $\mathcal{S}_0$. For $\varepsilon \neq 0$ we can express system (1) in the general form given in (2), namely

$$(\dot{x}, \dot{y}) = \begin{cases} \overline{F}(x, y, \varepsilon) & \text{if} \quad h(x, y, \varepsilon) \leq 0, \\ \overline{G}(x, y, \varepsilon) & \text{if} \quad h(x, y, \varepsilon) \geq 0, \end{cases} \tag{5}$$

where $\overline{F}(x,y,\varepsilon) = (F, H/\varepsilon)$ and $\overline{G}(x,y,\varepsilon) = (G, H/\varepsilon)$. We will use the notation $Z_\varepsilon = (\overline{F}, \overline{G})$ to represent the Filippov slow-fast system and Z_0 to represent the reduced problem. We say that system (1) is *locally simple* at $p = (x_0, y_0, 0) \in \mathbb{R}^{n+1} \times \mathbb{R}$ if one of the following conditions is satisfied:

a) $\frac{\partial h}{\partial x}(p) \neq 0$ and $h(x_0, y, 0) = 0$, for all y close to y_0, or;

b) there exists a neighborhood U of (x_0, y_0) in $\mathbb{R}^{n+1}$ such that $\frac{\partial H}{\partial x}(q) = 0$, for all $q \in U \cap \mathcal{M}_\varepsilon$.

2 Fold, Cusp and Hyperbolic Singularities

We say that $p \in \mathcal{M}$ is a *fold–regular* singularity of (2) provided $Fh(p) = 0$ and $F^2h(p) \neq 0$ and $Gh(p) \neq 0$ or $Gh(p) = 0$ and $G^2h(p) \neq 0$ and $Fh(p) \neq 0$. Moreover:

i) In the first case, we say that the fold–regular singularity $p \in \mathcal{M}$ is visible if $F^2h(p) < 0$ and invisible if $F^2h(p) > 0$.

ii) In the second case, it is visible provided $G^2h(p) > 0$ and invisible provided $G^2h(p) < 0$.

Let $p \in \mathcal{M}$ be a *fold–fold* singularity of (2), i.e., both vector fields F and G have a fold or quadratic tangency at the same point $p \in \mathcal{M}$. We distinguish the following cases:

i) **Elliptic case:** $F^2h(p) > 0$ and $G^2h(p) < 0$ (invisible two–fold).

ii) **Parabolic case:** $F^2h(p) < 0$ and $G^2h(p) < 0$ (visible fold – invisible fold) or $F^2h(p) > 0$ and $G^2h(p) > 0$ (invisible fold – visible fold).

iii) **Hyperbolic case:** $F^2h(p) < 0$ and $G^2h(p) > 0$ (visible two–fold).

Theorem 2.1. *Let $Z_\varepsilon(x,y)$ be a C^r family defined by (1), with $r \geq 2$. Consider $\overline{p} = (p, f_0(p), 0) \in \mathcal{S}_0 \cap \mathcal{M}_0$ a fold–regular singularity of the reduced problem Z_0 and suppose that Z_ε is locally simple at $\overline{p}$. Then there exists $\varepsilon_1 > 0$ such that:*

(i) There is a C^{r-1} family $\{\overline{p}_\varepsilon : \varepsilon \in (-\varepsilon_1, \varepsilon_1)\}$ such that $\overline{p}_0 = \overline{p}$ and $\overline{p}_\varepsilon$ is a fold–regular singularity of Z_ε;

(ii) If $\overline{p}$ is a visible (resp. invisible) fold–regular of Z_0 then $\overline{p}_\varepsilon$ is a visible (resp. invisible) fold–regular of Z_ε.

Proof. Suppose that the system is locally simple at $\overline{p} = (p, f_0(p), 0) \in \mathcal{S}_0 \cap \mathcal{M}_0$. Then, the following relations are valid

$$\tilde{F}^i\tilde{h}(p) = F^i h(p, f_0(p), 0) \quad \text{and} \quad \tilde{G}^i\tilde{h}(p) = G^i h(p, f_0(p), 0),$$

for all $i \in \mathbb{N}$.

We suppose that $\overline{p} = (p, f_0(p), 0)$ is a fold for the vector field $\tilde{F}$ and regular for $\tilde{G}$. The opposite case is similar. The fold–regular point $\overline{p}$ satisfies the following conditions

$$\tilde{F}\tilde{h}(p) = 0, \quad \tilde{F}^2\tilde{h}(p) \neq 0 \quad \text{and} \quad \tilde{G}\tilde{h}(p) \neq 0.$$

Since Z_ε is locally simple at $\overline{p}$ these conditions are equivalent to

$$Fh(\overline{p}) = 0, \quad F^2h(\overline{p}) \neq 0 \quad \text{and} \quad Gh(\overline{p}) \neq 0.$$

On the other hand, for each $\varepsilon \neq 0$ small, the vector field $\overline{F}(x,y,\varepsilon) = (F, H/\varepsilon)$ has a tangency at a point q if, and only, if such point satisfies the equation $\overline{F}h(x,y,\varepsilon) = 0$, that is equivalent to

$$Fh(x,y,\varepsilon) + \frac{\partial h}{\partial y}\frac{H}{\varepsilon}(x,y,\varepsilon) = 0. \tag{6}$$

DOI: 10.5540/03.2015.003.01.0002

In order to obtain a family $\overline{p}_\varepsilon$ of tangency points such that $\overline{p}_0 = \overline{p}$ we need to solve the equation (6) restricted to manifold $y = f_\varepsilon(x)$. Since $H(x, f_\varepsilon(x), \varepsilon) = 0$, it is enough to solve the following equation

$$Fh(x, f_\varepsilon(x), \varepsilon) = 0.$$

We have that $F^2h(p, f_0(p), 0) \neq 0$. But, $F^2h = F(Fh) = \frac{\partial(Fh)}{\partial x} \cdot F$, so, in particular,

$$\frac{\partial(Fh)}{\partial x}(p, f_0(p), 0) \neq 0.$$

By the Implicit Function Theorem, for each $\varepsilon \neq 0$ sufficiently small, there exists a unique $x = x(\varepsilon)$ such that $Fh(x(\varepsilon), f_\varepsilon(x(\varepsilon)), \varepsilon) = 0$. Take $\overline{p}_\varepsilon = (x(\varepsilon), f_\varepsilon(x(\varepsilon)), \varepsilon)$. Thus, $\overline{p}_\varepsilon$ is a family of tangency points of Z_ε such that $\overline{p}_0 = \overline{p}$.

For $\overline{p}_\varepsilon$ be a fold–regular singularity of Z_ε, we need to prove that

$$\left[F \cdot \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial x} \cdot F + \frac{\partial h}{\partial y}\frac{H}{\varepsilon}\right) + \frac{H}{\varepsilon}\frac{\partial}{\partial y}\left(\frac{\partial h}{\partial x} \cdot F + \frac{\partial h}{\partial y}\frac{H}{\varepsilon}\right)\right](\overline{p}_\varepsilon) \neq 0$$

and

$$\left(\frac{\partial h}{\partial x} \cdot G + \frac{\partial h}{\partial y}\frac{H}{\varepsilon}\right)(\overline{p}_\varepsilon) \neq 0.$$

As before, since we are restricted to manifold $y = f_\varepsilon(x)$ and $H(x, f_\varepsilon(x), \varepsilon) = 0$, these two conditions are summarized to

$$F^2h(\overline{p}_\varepsilon) \neq 0 \quad \text{and} \quad Gh(\overline{p}_\varepsilon) \neq 0.$$

As $\overline{p} = (p, f_0(p), 0)$ is a fold–regular singularity of Z_0 we have that $\tilde{F}^2\tilde{h}(p) = F^2h(p, f_0(p), 0) \neq 0$ and $\tilde{G}\tilde{h}(p) = Gh(p, f_0(p), 0) \neq 0$. By continuity of the functions F^2h and Gh, follows that $F^2h(q) \neq 0$ and $Gh(q) \neq 0$ for all q in a given neighborhood of $\overline{p}$. In particular, for each $\varepsilon \neq 0$ sufficiently small, $F^2h(\overline{p}_\varepsilon) \neq 0$ and $Gh(\overline{p}_\varepsilon) \neq 0$, i.e., $\overline{p}_\varepsilon$ is a fold–regular singularity of Z_ε.

In order to prove the item (ii), suppose that the fold–regular $\overline{p}$ is visible. So, $\tilde{F}^2\tilde{h}(p) = F^2h(p, f_0(p), 0) < 0$. Again, by continuity, we can conclude that $F^2h(\overline{p}_\varepsilon) < 0$, for each $\varepsilon \neq 0$ sufficiently small, i.e., $\overline{p}_\varepsilon$ is a visible fold–regular singularity of Z_ε. Similarly, if $\overline{p}$ is an invisible fold–regular of Z_0, then $\overline{p}_\varepsilon$ will also be an invisible fold–regular of Z_ε. ■

We investigated the effect of singular perturbations at tangency points. We can prove that singularities of the kind fold are robust with respect to singular perturbations. On the other hand, cusp singularities are not robust with respect to singular perturbations. We also study the unfolding of cusp singularities and hyperbolic equilibria. In short, with the same techniques used in Theorem 2.1, we can prove that:

- for any $n \geq 2$, fold–regular singularities are persistent;
- for $n = 2$, cusp–regular singularities are not persistent but for $n \geq 3$ they are so;
- for $n = 2$, fold–fold singularities are not persistent but for $n \geq 3$ they are so;
- for $n = 2, 3$, fold–cusp singularities are not persistent but for $n \geq 4$ they are so;
- for $n = 2$, hyperbolic/equilibrium–regular singularities are not persistent.

3 Singularly Perturbed Non-Smooth Systems

First of all, we introduce the notation.

$$S_\varepsilon: \quad \dot{x} = \begin{cases} F^-(x, y, \varepsilon), & \text{if} \quad h(x, y) \leq 0 \\ F^+(x, y, \varepsilon), & \text{if} \quad h(x, y) \geq 0 \end{cases} \qquad \varepsilon\dot{y} = G(x, y, \varepsilon), \tag{7}$$

where $x = (x_1, .., x_n) \in \mathbb{R}^n$, $y \in \mathbb{R}$, ε is a small positive parameter, $F^{\mp}$, G and h are C^r functions, with $r \geq 1$. The second one is given by system of kind

$$S^*_\varepsilon : \quad \varepsilon\dot{x} = \begin{cases} f^-(x, y, \varepsilon), & \text{if} \quad h(x, y) \leq 0 \\ f^+(x, y, \varepsilon), & \text{if} \quad h(x, y) \geq 0 \end{cases} \qquad \dot{y} = g(x, y, \varepsilon), \tag{8}$$

where $x \in \mathbb{R}$, $y \in \mathbb{R}^n$, $f^{\mp}$, $g = (g_1, ..., g_n)$ and h are C^r functions, with $r \geq 1$.

The *singularly perturbed non-smooth systems* which we consider here are systems given on the forms S_ε or S^*_ε. For $\varepsilon = 0$ in S_ε and S^*_ε, we have the *reduced systems* given respectively by

$$S_0 : \quad \dot{x} = \begin{cases} F^-(x, y, 0), & \text{if} \quad x_1 \leq 0 \\ F^+(x, y, 0), & \text{if} \quad x_1 \geq 0 \end{cases} \qquad 0 = G(x, y, 0).$$

$$S^*_0 : \quad 0 = \begin{cases} f^-(x, y, 0), & \text{if} \quad x_1 \leq 0 \\ f^+(x, y, 0), & \text{if} \quad x_1 \geq 0 \end{cases} \qquad \dot{y} = g(x, y, 0).$$

Denote $\mathcal{M}_\varepsilon = \{(x, y) : G(x, y, \varepsilon) = 0\}$ and $\mathcal{M}^{\mp}_\varepsilon = \{(x, y) : f^{\mp}(x, y, \varepsilon) = 0\}$. The sets $\mathcal{M}_0$, $\mathcal{M}^-_0$ and $\mathcal{M}^+_0$ are called *critical manifolds*. The timescale $\tau = t/\varepsilon$ transforms the systems S_ε and S^*_ε in the systems

$$x' = \begin{cases} \varepsilon F^-(x, y, \varepsilon), & \text{if} \quad x_1 \leq 0 \\ \varepsilon F^+(x, y, \varepsilon), & \text{if} \quad x_1 \geq 0 \end{cases} \qquad y' = G(x, y, \varepsilon). \tag{9}$$

$$x' = \begin{cases} f^-(x, y, \varepsilon), & \text{if} \quad x_1 \leq 0 \\ f^+(x, y, \varepsilon), & \text{if} \quad x_1 \geq 0 \end{cases} \qquad y' = \varepsilon g(x, y, \varepsilon). \tag{10}$$

We say that a compact $K \subset \mathcal{M}_0$ is normally hyperbolic if the real parts of the eigenvalues of $G_y(x, y, 0)$ are nonzero, for all $(x, y) \in K$. The system S_ε and S^*_ε are called *slow systems* and the systems (9) and (10) are called *fast systems*. In both cases, for $\varepsilon > 0$, the slow and fast systems have the same phase-portrait. For $\varepsilon = 0$ the systems (9) and (10) are called *layer problems*. We can note that S_0 is defined in $\mathcal{M}_0$ which has a $(n-1)$- dimensional switching manifold represented by $\Sigma_r = \Sigma \cap \mathcal{M}_0$ and the crossing, sliding and escaping regions are given by $\Sigma^c_r = \Sigma^c \cap \mathcal{M}_0$, $\Sigma^s_r = \Sigma^s \cap \mathcal{M}_0$ and $\Sigma^e_r = \Sigma^e \cap \mathcal{M}_0$. If we suppose

$$\begin{aligned} \forall (x, y) \in \mathcal{M}_0 : \quad & D_y G(x, y, 0) \neq 0, \\ \forall (x, y) \in \mathcal{M}^-_0 \cup \mathcal{M}^+_0 : \quad & D_x f^{\mp}(x, y, 0) \neq 0, \end{aligned}$$

then $y = \psi(x)$ locally parametrize $\mathcal{M}_0$ by solving $G(x, y, 0) = 0$ and $x = \xi^{\mp}(y)$ locally parametrize $\mathcal{M}^-_0 \cup \mathcal{M}^+_0$ by solving $f^{\mp}(x, y, 0) = 0$.

In what follows, we define the regularization for systems S_0, S^*_0, S_ε and S^*_ε. The regularization of systems S_0 and S^*_0 are defined respectively by

$$\begin{aligned} S_{\lambda,0} \quad & : \quad \dot{x} = \frac{F^+ + F^-}{2} + \varphi\left(\frac{x_1}{\lambda}\right)\frac{F^+ - F^-}{2}, \quad G = 0, \\ S^*_{\lambda,0} \quad & : \quad \frac{f^+ + f^-}{2} + \varphi\left(\frac{x}{\lambda}\right)\frac{f^+ - f^-}{2} = 0, \quad \dot{y} = g, \end{aligned}$$

where φ is a transition function and the other functions are valued at $(x, y, 0)$. The regularizations of systems S_ε and S^*_ε are the 2-parameters system families $S_{\lambda,\varepsilon}$ and $S^*_{\lambda,\varepsilon}$ defined respectively by

$$\begin{aligned} S_{\lambda,\varepsilon} \quad & : \quad \dot{x} = \frac{F^+ + F^-}{2} + \varphi\left(\frac{x_1}{\lambda}\right)\frac{F^+ - F^-}{2}, \quad \varepsilon\dot{y} = G, \\ S^*_{\lambda,\varepsilon} \quad & : \quad \varepsilon\dot{x} = \frac{f^+ + f^-}{2} + \varphi\left(\frac{x}{\lambda}\right)\frac{f^+ - f^-}{2}, \quad \dot{y} = g, \end{aligned}$$

where φ is a transition function and the other functions are valued at (x, y, ε).

4 Closed Poly–trajectories

Definition 4.1. *Consider $n = 2$.*

*(i) A curve Γ is a **closed poly-trajectory** if Γ is closed and the following conditions are satisfied.*

- *Γ contains arcs of at lest two of $X^-|_{\Sigma^-}$, $X^+|_{\Sigma^+}$ and X^Σ.*
- *The transition between arcs of X^- and arcs of X^+ happens in crossing points (and vice versa).*
- *The transition between arcs of X^- (or X^+) and arcs of X^Σ happens through either fold points or regular points in the sliding or escaping region, respecting orientation.*

(ii) Let Γ be a closed poly-trajectory. We say that

- *Γ is a **crossing poly-trajectory** if Γ meets Σ just in crossing points.*
- *Γ is a **sliding poly-trajectory** if Γ contains at least a fold point.*

*(iii) Let Γ be a closed poly-trajectory. We say that Γ is **hyperbolic** if*

- *Γ is a crossing poly-trajectory and $\eta'(p) \neq 1$ where η is the first return map defined on a segment N with $p \in N \pitchfork \gamma$;*
- *Γ is a sliding poly-trajectory and all arcs of F^Σ are sliding or all are escaping.*

Theorem 4.1. *Fix $n = 2$. Suppose that S_0 has a hyperbolic crossing or sliding poly-trajectory Γ_0.*

(a) For small $\lambda > 0$ the regularized system $S_{\lambda,0}$ has a hyperbolic limit cycle $\Gamma_{\lambda,0}$, such that $\Gamma_{\lambda,0} \to \Gamma_0$ when $\lambda \to 0$, according to Hausdorff distance.

(b) If Γ_0 is normally hyperbolic, then for small $\varepsilon, \lambda > 0$, the regularized system $S_{\lambda,\varepsilon}$ has a hyperbolic limit cycle $\Gamma_{\lambda,\varepsilon}$, such that $\Gamma_{\lambda,\varepsilon} \to \Gamma_0$, when $(\lambda, \varepsilon) \to (0, 0)$, according to Hausdorff distance.

(c) If Γ_0 is normally hyperbolic then the non-smooth system S_ε has a closed poly-trajectory Γ_ε, for small values of $\varepsilon > 0$.

Theorem 4.2. *Consider system S^*_ε satisfying $h(x, y) = x$, $f^-(0, y, 0) = f^+(0, y, 0)$ and $f^-(0, y, \varepsilon) \neq f^+(0, y, \varepsilon)$, for any $\varepsilon > 0$. Let $P = (0, \bar{y})$ be an equilibrium point of reduced problem S^*_0 on Σ, i.e., $g(0, \bar{y}, 0) = 0$.*

(a) If there exist $\mathcal{C} \subset \mathbb{R}^n$ neighborhood of $\bar{y}$ and small $\varepsilon_0 > 0$ such that $f^-(0, y, \varepsilon)f^+(0, y, \varepsilon) < 0$ for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, then there exists a continuous family P_ε such that

- *P_ε is an equilibrium point of the sliding vector field associated to system S^*_ε;*
- *$P_0 = P$;*
- *P_0 is an equilibrium point of the regularized system $S^*_{\lambda,0}$;*
- *the sliding vector field around P_ε is topologically equivalent to*

$$B(0, \bar{y}, 0) = \begin{pmatrix} g^1_{y_1}(0, \bar{y}, 0) & \dots & g^1_{y_n}(0, \bar{y}, 0) \\ \vdots & \vdots & \vdots \\ g^n_{y_1}(0, \bar{y}, 0) & \dots & g^n_{y_n}(0, \bar{y}, 0) \end{pmatrix}.$$

(b) There exist a small $\varepsilon_1 > 0$ and a continuous family Q_ε, with $\varepsilon \in (-\varepsilon_1, \varepsilon_1)$ such that

- $Q_0 = P$;
- Q_ε *is an equilibrium point of the regularized system* $S^*_{\lambda,\varepsilon}$.

Now, we add a discontinuity at function g and we suppose that h does not depend of variable x. We get the following system

$$\varepsilon\dot{x} = \begin{cases} f^-(x,y,\varepsilon), & \text{if} \quad h(y) \leq 0 \\ f^+(x,y,\varepsilon), & \text{if} \quad h(y) \geq 0 \end{cases}, \ \dot{y} = \begin{cases} g^-(x,y,\varepsilon), & \text{if} \quad h(y) \leq 0 \\ g^+(x,y,\varepsilon), & \text{if} \quad h(y) \geq 0. \end{cases} \tag{11}$$

We suppose that the condition

$$f^-(x,y,\varepsilon) = f^+(x,y,\varepsilon), \forall (x,y) \in \Sigma = \{h = 0\}, \forall \varepsilon \geq 0$$

is satisfied and $f^{\mp}(x,y,\varepsilon) = 0$ can be solved by $x = \xi^{\mp}_\varepsilon(y)$, locally, for all $\varepsilon \geq 0$. Thus, the reduced system becomes a non-smooth system given by

$$\dot{y} = \begin{cases} g^-(\xi^-_0(y), y, 0), & \text{if} \quad h(y) \leq 0 \\ g^+(\xi^+_0(y), y, 0), & \text{if} \quad h(y) \geq 0 \end{cases}. \tag{12}$$

We say that a point $p \in \mathcal{M}^-_0 \cap \mathcal{M}^+_0 \cap \Sigma$ satisfy the *propriety* $\mathcal{P}$ if there exists a neighborhood V of p, such that

$$f^-(q,\varepsilon) = f^+(q,\varepsilon) \quad \text{and} \quad f^{\mp}_y(q) = 0, \quad \forall q \in V \cap \mathcal{M}^-_\varepsilon \cap \mathcal{M}^+_\varepsilon.$$

Theorem 4.3. *Let* $\mathcal{N} \subset \mathcal{M}_0$ *be a normally hyperbolic equilibrium point or periodic orbit of the reduced system* (12) *with a* j^s*-dimensional local stable manifold* W^s *and a* j^u*-dimensional local unstable manifold* W^u*. Suppose that* $\mathcal{N}$ *satisfies the propriety* $\mathcal{P}$*. Thus, there exists an* ε*-continuous family* $\mathcal{N}_\varepsilon$ *such that*

(i) $\mathcal{N}_0 = \mathcal{N}$;

(ii) $\mathcal{N}_\varepsilon$ *is a hyperbolic equilibrium point or periodic orbit of sliding vector field associated to system* (11)*, with a* (j^s+k^s)*-dimensional local stable manifold* $\mathcal{N}^s_\varepsilon$ *and a* (j^u+k^u)*-dimensional local unstable manifold* $\mathcal{N}^s_\varepsilon$.

Referências

[1] P.T.Cardin, P.R. da Silva, M.A. Teixeira, On singularly perturbed Filippov systems, *http://www.mat.feis.unesp.br/arquivos/pedro/Preprint.pdf*, (2011).

[2] N.Fenichel , Geometric singular perturbation theory for ordinary differential equations, *J. Diff. Equations* **31** (1979), 53–98.

[3] A. F. Filippov, Differential Equations with Discontinuous Righthand Sides, Mathematics and its Applications (Soviet Series), Kluwer Academic Publishers, Dordrecht (1988).

[4] P.Szmolyan, Transversal Heteroclinic and Homoclinic Orbits in Singular Perturbation Problems, *J. Diff. Equations* **92** (1990), 252–281.

