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An Approach for Solving Interval Optimal Control Problems

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Abstract. In this work, it will be considered optimal control problems in which the objective function is interval-valued. The concept of optimality will be defined through the lower-upper order relation (LU-order). Problem data is assumed to satisfy merely Lipschitz continuity. Necessary optimality conditions in the form of a maximum principle are obtained.

Keywords. Multi-objective Optimization Problems, Optimality Conditions, Maximum Principle, Interval Optimal Control Problems, $LU-$order relation, $LU-$processes.

1 Introduction

Optimal control theory play an important role in a lot of problems where the objective is to describe the "controls" that will cause a process to satisfy aerospace engineering, biological, chemical, computational, economical, medical, physical, or social constraints and at the same time these "controls" have to minimize some criterion. Many authors have studied optimal control problems from different points of view and can be found in many textbooks (for example, see M. Athans and P. L. Falb, [1], and E. R. Pinch [5]). Optimal control problems are usually solved with the pontryagin maximum principle (PMP) (see R. Vinter, [7]), which is a generalization of the classic Euler–Lagrange and Weierstrass necessary optimality conditions for the calculus of variations, and here is not going to be different. The goal of this work is to give necessary optimality conditions for interval optimal control problems via classical bi-objective optimal control problems.

Given $v, w \in \mathbb{R}^n$, we denote the usual inner product between v and w as $v \cdot w$.

By $v \leqq w$ we mean $v_i \leq w_i$ for all $i \in \{1, 2, \ldots, n\}$; by $v \leq w$ we mean $v_i \leq w_i$ for all $i \in \{1, 2, \ldots, n\}$ and $v \neq w$; by $v < w$ we mean $v_i < w_i$ for all $i \in \{1, 2, \ldots, n\}$.

$\mathcal{L}$ denotes the *Lebesgue subsets* of a given interval $[a, b]$; $\mathcal{B}^m$ denotes the *Borel sets* of $\mathbb{R}^m$; and $\mathcal{L} \times \mathcal{B}^m$ denotes the product $\sigma-$algebra.

Given a multifunction $U : [a, b] \rightrightarrows \mathbb{R}^n$, $\mathrm{Gr}\,(U)$ means the *graph* of U.

The space of the *absolutely continuous functions* is denoted by $W^{1,1}([a, b]; \mathbb{R}^n)$.

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Given a closed set $S \subset \mathbb{R}^n$ and a point $x \in S$, the set of all directions $v \in \mathbb{R}^n$ such that there exists $M > 0$ satisfying $v \cdot (y-x) \leq M\|y-x\|^2$ for all $y \in S$, is said to be the *proximal normal cone to* S *at* x, denoted by $N_S^P(x)$. The set of all directions $v \in \mathbb{R}^n$ such that there exist sequences $x_i \xrightarrow{S} x$ and $v_i \to v$ satisfying $v_i \in N_S^P(x_i)$ for all i, is said to be the *limiting normal cone to* S *at* x, denoted by $N_S(x)$.

Let $\gamma : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous function and $x \in \operatorname{dom}\gamma$. $\partial\gamma(x)$ is the *limiting Mordukhovich's subdifferential of* γ *at* x defined as the set $\partial\gamma(x) = \{\zeta \,|\, (\zeta, -1) \in N_{\operatorname{epi}\gamma}(x, \gamma(x))\}$.

One of the main tools used in this work is the maximum principle for the following *multi-objective optimal control problem*:

$$(MCP) \begin{cases} \text{minimize} \ \ g(x(a), x(b)) \\ \text{subject to } x'(t) = \gamma(t, x(t), u(t)) \text{ a.e. } t \in [a,b] \\ \qquad\qquad\quad (x(a), x(b)) \in S, \\ \qquad\qquad\quad u(t) \in U(t) \text{ a.e. } t \in [a,b], \end{cases}$$

where $g = \begin{bmatrix} g_1 & g_2 & \dots & g_k \end{bmatrix}^\top : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^k$, $\gamma : [a,b] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, S is a closed subset of $\mathbb{R}^n \times \mathbb{R}^n$, $U : [a,b] \rightrightarrows \mathbb{R}^n$, a and b are fixed.

A measurable function $u : [a,b] \to \mathbb{R}^m$ such that $u(t) \in U(t)$ for a.e. $t \in [a,b]$ is said to be a *control function*.

A pair (x,u) consisting of $x \in W^{1,1}([a,b]; \mathbb{R}^n)$ and u obeying the differential equation above is called a *process*.

(x,u) is an *admissible process* if x corresponds to $u \in U$ which satisfies $(x(a), x(b)) \in S$.

An admissible process (x^*, u^*) is a *Pareto optimal process* if there exists no other admissible process (x,u) such that $g(x(a), x(b)) \leq g(x^*(a), x^*(b))$.

An admissible process (x^*, u^*) is a *weak Pareto optimal process* if there exists no other admissible process (x,u) such that $g(x(a), x(b)) < g(x^*(a), x^*(b))$.

Let (x^*, u^*) be an admissible process of (MCP). For some $\delta > 0$, the following are satisfied:

(H1) The function $\gamma(\cdot, x, \cdot)$ is $\mathcal{L} \times \mathcal{B}^m$ measurable, for each $x \in \mathbb{R}^n$;

(H2) There exists a $\mathcal{L} \times \mathcal{B}^m$ measurable function $l : [a,b] \times \mathbb{R}^m \to \mathbb{R}$ such that $t \mapsto l(t, u^*(t))$ is integrable, and for a.e. $t \in [a,b]$, $\|\gamma(t,x,u) - \gamma(t,\breve{x},u)\| \leq l(t,u)\|x - \breve{x}\|$ for all $x, \breve{x} \in x^*(t) + \delta\mathbb{B}$ and for all $u \in U(t)$, where $\mathbb{B} := \mathbb{B}(\boldsymbol{0}, 1) = \{x \in \mathbb{R}^n \,|\, \|x\| < 1\}$ denotes the open unit ball in $\mathbb{R}^n$;

(H3) $\operatorname{Gr}(U)$ is $\mathcal{L} \times \mathcal{B}^m$ measurable;

(H4) g is locally Lipschitz continuous.

Let $\mathcal{H} : [a,b] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ denote the *Unmaximized Hamiltonian function* $\mathcal{H}(t,x,p,u) := p \cdot \gamma(t,x,u)$.

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Theorem 1.1 (Pontryagin Maximum Principle, see V. A. de Oliveira and G. N. Silva [2]). *If* (x^*, u^*) *is a weak Pareto optimal process of* (MCP)*, then there exist a scalar* λ *(equal to* 0 *or* 1*), a nonzero vector* $\omega \in \mathbb{R}^k$*, and* $p \in W^{1,1}([a,b],\mathbb{R}^n)$ *such that*

(1) $-p'(t) \in \text{co}\,\{\partial_x \mathcal{H}(t, x^*(t), p(t), u^*(t))\}$ *a.e.* $t \in [a,b]$,

(2) $(p(a), -p(b)) \in \lambda\partial(\omega \cdot g)(x^*(a), x^*(b)) + N_S(x^*(a), x^*(b))$,

(3) $\max\limits_{u\in U(t)} \mathcal{H}(t, x^*(t), p(t), u) = \mathcal{H}(t, x^*(t), p(t), u^*(t))$ *a.e.* $t \in [a,b]$,

(4) $\|p\| + \lambda > 0$, $\|\omega\| = 1$, $\omega \geqq \mathbf{0}$.

Now, consider the space $\mathcal{K}_C = \{[\underline{a}, \overline{a}] \,|\, \underline{a}, \overline{a} \in \mathbb{R} \text{ and } \underline{a} \leq \overline{a}\}$. Given $A = [\underline{a}, \overline{a}], B = [\underline{b}, \overline{b}], C = [\underline{c}, \overline{c}] \in \mathcal{K}_C$ and $\lambda \in \mathbb{R}$, the interval arithmetic operations are defined by

$$A + B = [\underline{a}, \overline{a}] + [\underline{b}, \overline{b}] = [\underline{a} + \underline{b}, \overline{a} + \overline{b}], \quad A\lambda = \begin{cases} [\underline{a}\lambda, \overline{a}\lambda] & \text{if } \lambda \geq 0 \\ [\overline{a}\lambda, \underline{a}\lambda] & \text{if } \lambda < 0 \end{cases}$$

and

$$A \ominus_{gH} B = C \Leftrightarrow \begin{cases} A = B + C & \text{if } \text{len}(B) \leq \text{len}(A) \text{ or} \\ B = A + (-1)C & \text{if } \text{len}(B) > \text{len}(A), \end{cases}$$

where $\text{len}(D)$ denotes the length of an interval $D = [\underline{d}, \overline{d}] \in \mathcal{K}_C$, i.e., $\text{len}(D) = \overline{d} - \underline{d}$. The $gH-$difference of two intervals always exists (see L. Stefanini and B. Bede, [6]) and

$$A \ominus_{gH} B = \left[\min\{\underline{a} - \underline{b}, \overline{a} - \overline{b}\}, \max\{\underline{a} - \underline{b}, \overline{a} - \overline{b}\}\right].$$

Definition 1.1 (See U. W. Kulish and W. L. Miranker, [3]). *Let* $A = [\underline{a}, \overline{a}]$ *and* $B = [\underline{b}, \overline{b}] \in \mathcal{K}_C$*. The order relation lower and upper, LU in short, is defined by*

1. $A \leqq_{LU} B$ *if and only if* $\underline{a} \leq \underline{b}$ *and* $\overline{a} \leq \overline{b}$.
2. $A \leq_{LU} B$ *if and only if* $A \leqq_{LU} B$ *and* $A \neq B$*, that is, either* $\underline{a} < \underline{b}$ *and* $\overline{a} \leq \overline{b}$ *or* $\underline{a} \leq \underline{b}$ *and* $\overline{a} < \overline{b}$.
3. $A <_{LU} B$ *if and only if* $\underline{a} < \underline{b}$ *and* $\overline{a} < \overline{b}$.

2 The Interval Optimal Control Problem $(IOCP)$

This work deals with the *interval-valued optimal control problem* posed as follows:

$$\begin{cases} \text{minimize} \;\; G(x(b)) = [\underline{g}(x(b)), \overline{g}(x(b))] \\ \text{subject to } x'(t) = \gamma(t, x(t), u(t)) \text{ a.e. } t \in [a,b], \\ \qquad\qquad (x(a), x(b)) \in \{x_a\} \times \mathbb{R}^n = S, \\ \qquad\qquad u(t) \in U(t) \text{ a.e. } t \in [a,b], \end{cases}$$

where $G : \mathbb{R}^n \to \mathcal{K}_C$ is given as $G(x(b)) := [\underline{g}(x(b)), \overline{g}(x(b))]$ with $\underline{g}, \overline{g} : \mathbb{R}^n \to \mathbb{R}$ and $\underline{g}(x(b)) \leq \overline{g}(x(b))$ for all trajectories x, $\gamma : [a,b] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $S := \{x_a\} \times \mathbb{R}^n$ is a closed subset of $\mathbb{R}^n \times \mathbb{R}^n$, $U : [a,b] \rightrightarrows \mathbb{R}^n$, a and b are fixed.

The set of admissible controls $\mathcal{U}_{ad}$ is given by

$$\mathcal{U}_{ad} = \{u \in \mathcal{M}([a,b];\mathbb{R}^m) \,|\, u(t) \in U(t) \ \text{ a.e. } t \in [a,b]\}.$$

The set of admissible trajectories $\mathcal{X}_{ad}$ is given by:

$$\mathcal{X}_{ad} = \{x \in W^{1,1}([a,b];\mathbb{R}^n) \,|\, x'(t) = \gamma(t, x(t), u(t)) \,\text{a.e.}\, t \in [a,b],\ (x(a), x(b)) \in S,\ u \in \mathcal{U}_{ad}\}.$$

Therefore, the interval optimal control problem that will be considered, is

$$(IOCP) \begin{cases} \text{minimize} \ \ G(x(b)) = [\underline{g}(x(b)), \overline{g}(x(b))] \\ \text{subject to } (x,u) \in \mathcal{X}_{ad} \times \mathcal{U}_{ad}. \end{cases}$$

From the order relation LU, it will be defined the optimal $LU-$processes for interval optimal control problems.

Definition 2.1. *Let (x^*, u^*) be an admissible process of $(IOCP)$, i.e., $(x^*, u^*) \in \mathcal{X}_{ad} \times \mathcal{U}_{ad}$.*

*(i) (x^*, u^*) is an **optimal** $LU-$**process** of $(IOCP)$ if there exists no $(x,u) \in \mathcal{X}_{ad} \times \mathcal{U}_{ad}$ such that $G(x(b)) \leq_{LU} G(x^*(b))$.*

*(ii) (x^*, u^*) is a **weak optimal** $LU-$**process** of $(IOCP)$ if there exists no $(x,u) \in \mathcal{X}_{ad} \times \mathcal{U}_{ad}$ such that $G(x(b)) <_{LU} G(x^*(b))$.*

Hypothesis

The Hamilton function, $\mathcal{H}$, for (IOCP) is exactly the one already defined for (MCP), since the dynamics are the same.

Definition 2.2. *Given $S \subseteq \mathbb{R}^n$ an open and nonempty set, let $F : S \to \mathcal{K}_C$ be an interval-valued function.*

(1) We say that F is Lipschitz continuous of rank K if we have that

$$d_H(F(x), F(y)) \leq K\|x - y\| \ \text{ for all } x, y \in S,$$

where d_H is the Pompeiu-Hausdorff distance[3] between $F(x)$ and $F(y) \in \mathcal{K}_C$.

(2) We say that F is Lipschitz continuous locally near a given point $x \in \mathbb{R}^n$ of rank K if for some $\epsilon > 0$, we have

$$d_H(F(y), F(z)) \leq K\|y - z\| \ \text{ for all } y, z \in \mathbb{B}(x, \epsilon),$$

where $\mathbb{B}(x, \epsilon) = \{x' \in \mathbb{R}^n \,|\, \|x - x'\| < \epsilon\}$ denotes the open ball of center x and radius ϵ in $\mathbb{R}^n$.

[3]The Pompeiu-Hausdorff distance between $A = [\underline{a}, \overline{a}]$ and $B = [\underline{b}, \overline{b}] \in \mathcal{K}_C$ denoted as $d_H : \mathcal{K}_C \times \mathcal{K}_C \to [0, +\infty)$ is given by $d_H(A,B) = \max\{|\underline{a} - \underline{b}|, |\overline{a} - \overline{b}|\}$.

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In [4], R. E. Moore, R. B. Kearfott and M. J. Cloud saw the last definition, but for the case $n = 1$.

Proposition 2.1. *Let $S \subseteq \mathbb{R}^n$ an open and nonempty set, let $F : S \to \mathcal{K}_C$ be an interval-valued function such that $F(x) = [\underline{f}(x), \overline{f}(x)]$, where $\underline{f}, \overline{f} : S \to \mathbb{R}$. Then F is Lipschitz continuous if and only if the extreme functions $\underline{f}$ and $\overline{f}$ are Lipschitz continuous.*

Proof. The proof is simple and will be omitted. □

Let (x^*, u^*) be an admissible process of $(IOCP)$. As before, for some $\delta > 0$, we are going to suppose that γ and U satisfy $(H1) - (H3)$ and also we are going to suppose that:

$(H4)'$ G is Lipschitz continuous locally.

3 Optimality Conditions for $LU-$processes

Necessary optimality conditions will be obtained, for optimal weak $LU-$processes of the interval optimal control problem $(IOCP)$.

Theorem 3.1. *If (x^*, u^*) is a weak optimal $LU-$process of $(IOCP)$, then (x^*, u^*) is a weak Pareto optimal process of the following classical bi-objective optimal control problem:*

$$(BCP)_{LU} \begin{cases} \textit{minimize} \quad g(x(b)) = (\underline{g}(x(b)), \overline{g}(x(b))) \\ \textit{subject to } (x, u) \in \mathcal{X}_{ad} \times \mathcal{U}_{ad}. \end{cases}$$

Conversely, if (x^*, u^*) is a weak Pareto optimal process of $(BCP)_{LU}$, then (x^*, u^*) is a weak optimal $LU-$process of $(IOCP)$.

Proof. If $(x^*, u^*) \in \mathcal{X}_{ad} \times \mathcal{U}_{ad}$ is a weak Pareto optimal process of $(BCP)_{LU}$, then there exists no $(x, u) \in \mathcal{X}_{ad} \times \mathcal{U}_{ad}$ such that

$$\begin{gathered} g(x(b)) < g(x^*(b)) \\ \Leftrightarrow (\underline{g}(x(b)), \overline{g}(x(b))) < (\underline{g}(x^*(b)), \overline{g}(x^*(b))) \\ \Leftrightarrow \underline{g}(x(b)) < \underline{g}(x^*(b)) \text{ and } \overline{g}(x(b)) < \overline{g}(x^*(b)) \\ \Leftrightarrow [\underline{g}(x(b)), \overline{g}(x(b))] <_{LU} [\underline{g}(x^*(b)), \overline{g}(x^*(b))] \\ \Leftrightarrow G(x(b)) <_{LU} G(x^*(b)), \end{gathered}$$

which is the definition of weak $LU-$optimality for (x^*, u^*). □

Theorem 3.2. *If $(x^*, u^*) \in \mathcal{X}_{ad} \times \mathcal{U}_{ad}$ is a weak optimal $LU-$process of $(IOCP)$, then there exist a multiplier $p^* \in W^{1,1}([a, b], \mathbb{R}^n)$, a scalar $\lambda^* \in \{0, 1\}$, and a nonzero vector $\omega^* = (\omega_1^*, \omega_2^*) \in \mathbb{R}^2$, such that for almost every $t \in [a, b]$,*

$$\begin{aligned} &(x^*)'(t) = \frac{\partial \mathcal{H}}{\partial p}(t, x^*(t), p^*(t), u^*(t)), \\ &-(p^*)'(t) \in \text{co } \{\partial_x \mathcal{H}(t, x^*(t), p^*(t), u^*(t))\}, \\ &x^*(a) = x_a \qquad \textit{and} \qquad -p^*(b) \in \lambda^* \omega_1^* \frac{\partial \underline{g}}{\partial x(b)}(x^*(b)) + \lambda^* \omega_2^* \frac{\partial \overline{g}}{\partial x(b)}(x^*(b)). \end{aligned}$$

Furthermore, the control u^* satisfies the following optimality condition for almost every $t \in [a,b]$

$$\mathcal{H}(t,x^*(t),p^*(t),u^*(t)) = \max_{u\in U(t)} \mathcal{H}(t,x^*(t),p^*(t),u).$$

Proof. Since (x^*,u^*) is a weak optimal $LU-$process, by Theorem 3.1, (x^*,u^*) is also a weak Pareto optimal process of $(BCP)_{LU}$, so that by Theorem 1.1, there exist a scalar $\lambda^* \in \{0,1\}$, a nonzero vector $\omega^* = (\omega_1^*,\omega_2^*) \in \mathbb{R}^2$ and $p^* \in W^{1,1}([a,b],\mathbb{R}^n)$, such that

(1) $-(p^*)'(t) \in \text{co}\,\{\partial_x \mathcal{H}(t,x^*(t),p^*(t),u^*(t))\}$ a.e. $t\in[a,b]$.

(2) $(p^*(a),-p^*(b)) \in \lambda^*\partial(\omega^*\cdot\widehat{G})(x^*(a),x^*(b)) + N_S(x^*(a),x^*(b))$, where $S=\{x_a\}\times\mathbb{R}$ and $\widehat{G}:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^2$ is defined by

$$\widehat{G}(x(a),x(b)) := (\underline{g}(x(b)),\overline{g}(x(b))).$$

Since

$$\omega^*\cdot\widehat{G}(x(a),x(b)) = (\omega_1^*,\omega_2^*)\cdot(\underline{g}(x(b)),\overline{g}(x(b))) = \omega_1^*\underline{g}(x(b)) + \omega_2^*\overline{g}(x(b)),$$

we have that

$$\partial(\omega^*\cdot\widehat{G})(x^*(a),x^*(b)) = \{\boldsymbol{0}\}\times\left[\omega_1^*\frac{\partial\underline{g}}{\partial x(b)}(x^*(b)) + \omega_2^*\frac{\partial\overline{g}}{\partial x(b)}(x^*(b))\right] \subseteq \mathbb{R}^{2n},$$

and since $(x^*,u^*)\in\mathcal{X}_{ad}\times\mathcal{U}_{ad}$ is a weak optimal $LU-$process of $(IOCP)$, then $x^*(a)=x_a$ and

$$N_S(x^*(a),x^*(b)) = N_{\{x_a\}\times\mathbb{R}^n}(x_a,x^*(b)) = N_{\{x_a\}}(x_a)\times N_{\mathbb{R}^n}(x^*(b)) = \mathbb{R}^n\times\{\boldsymbol{0}\}.$$

Therefore,

$$\begin{aligned}
(p^*(a),-p^*(b)) &\in \lambda^*\partial(\omega^*\cdot\widehat{G})(x^*(a),x^*(b)) + N_S(x^*(a),x^*(b))\\
&= \lambda^*\left(\{\boldsymbol{0}\}\times\left[\omega_1^*\frac{\partial\underline{g}}{\partial x(b)}(x^*(b)) + \omega_2^*\frac{\partial\overline{g}}{\partial x(b)}(x^*(b))\right]\right) + \mathbb{R}^n\times\{\boldsymbol{0}\}\\
&= \{\boldsymbol{0}\}\times\left[\lambda^*\omega_1^*\frac{\partial\underline{g}}{\partial x(b)}(x^*(b)) + \lambda^*\omega_2^*\frac{\partial\overline{g}}{\partial x(b)}(x^*(b))\right] + \mathbb{R}^n\times\{\boldsymbol{0}\}\\
&= \mathbb{R}^n\times\left[\lambda^*\omega_1^*\frac{\partial\underline{g}}{\partial x(b)}(x^*(b)) + \lambda^*\omega_2^*\frac{\partial\overline{g}}{\partial x(b)}(x^*(b))\right].
\end{aligned}$$

Then:

$$p^*(a) \in \mathbb{R}^n$$

and

$$-p^*(b) \in \lambda^*\omega_1^*\frac{\partial\underline{g}}{\partial x(b)}(x^*(b)) + \lambda^*\omega_2^*\frac{\partial\overline{g}}{\partial x(b)}(x^*(b)).$$

(3) $\max_{u\in U(t)}\mathcal{H}(t,x^*(t),p^*(t),u) = \mathcal{H}(t,x^*(t),p^*(t),u^*(t))$ a.e. $t\in[a,b]$.

(4) $\|p^*\| + \lambda^* > 0$, $\|\omega^*\| = 1$, $\omega^* \geq \boldsymbol{0}$.

Moreover, since $\mathcal{H}(t, x(t), p(t), u(t)) = p^\top(t)\gamma(t, x(t), u(t))$,

$$\frac{\partial \mathcal{H}}{\partial p}(t, x^*(t), p^*(t), u^*(t)) = \gamma(t, x^*(t), u^*(t)) = (x^*)'(t).$$

□

4 Conclusion

We considered an interval optimal control problem and we used the partial order relation *LU* for defining the optimal processes. Then, we presented a method to determine the optimal *LU*−processes. Using the Lipschitz concept for interval-valued functions and the *Maximum Principle* for classical multi-objective optimal control problems, we obtained necessary optimality conditions for interval optimal control problems.

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