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# A Methodology to Find Relations between Invariants of $n$ Symmetric Second-Order Tensors

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**Resumo**. A methodology is proposed to find either implicit or explicit relations between invariants in a minimal integrity basis for $n$ symmetric second-order tensors defined on a three-dimensional euclidean space. The implicit relations are called syzygies. In particular, the methodology i) allows the construction of a set of $6\,n-3$ independent invariants, ii) yields explicit non-polynomial expressions for certain invariants in terms of the remaining invariants in the integrity basis, and iii) allows the construction of syzygies. The results of this investigation are important in the modeling of biological structures, which, in general, are non-homogeneous and made of anisotropic viscoelastic materials that are subjected to large deformations.

**Palavras-chave**. Mechanics of Materials, Biological Structure, Response Function, Second-Order Tensor, Syzygy

## 1 Introduction

The application of invariance principles in continuum mechanics leads to the proposition of constitutive relations that depend on a list of invariants of physical variables, such as vectors and second-order tensors. Given a group of transformations acting on these variables, the central problem of the associated theory of invariants is to find a list of invariants from which all the other invariants can be generated without having redundant members. In the context of this work, where the invariants are polynomials of their arguments, we call this list an integrity basis if any polynomial invariant can be expressed as a polynomial of the members in the list. The integrity basis is minimal if it contains the smallest possible number of members. These members may, however, satisfy polynomial relations called syzygies, which do not allow any one invariant in these relations to be expressed in terms of the other invariants.

The construction of minimal integrity bases in continuum mechanics has been the subject of intense investigation since the 1950s, an account of which can be found in

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Spencer [7], and is, by now, well established. The determination of the number of syzygies in a minimal integrity basis together with the construction of these syzygies is, however, an active area of research. See, for instance, Shariff [5] and Shariff *et al.* [6].

Spencer [7] has constructed the minimum integrity basis for a finite number $n$ of symmetric tensors defined on the three-dimensional euclidean space, but has not determined the number of independent invariants of this basis. In this work we propose a methodology that i) allows the construction of a set of $6\,n-3$ independent invariants, ii) yields explicit non-polynomial expressions for certain invariants in terms of the remaining invariants in the integrity basis, and iii) allows the construction of syzygies. These results have important applications in modeling the mechanical response of viscoelastic polymer composites ( [3]), fiber-reinforced composites ( [1]), and biomaterials ( [2]).

## 2 The Methodology

Let us consider that all the symmetric tensors in the set $\{\mathbf{A}^{(1)}, \mathbf{A}^{(2)}, \ldots, \mathbf{A}^{(n)}\}$, where $n \geq 1$ have three distinct eigenvalues and that any two tensors in this set do not have parallel eigenvectors. If $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is the set of eigenvectors of $\mathbf{A}^{(1)}$ with associated eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$, we write

$$\mathbf{A}^{(1)} = \sum_{i=1}^{3} \lambda_i \, \mathbf{e}_i \otimes \mathbf{e}_i \,, \qquad \mathbf{A}^{(r)} = \sum_{i=1}^{3} \sum_{j=1}^{3} \beta_{ij}^{(r)} \mathbf{e}_i \otimes \mathbf{e}_j \qquad \text{for } r = 2, \ldots, n \,, \tag{1}$$

where $\beta_{ij}^{(r)} \stackrel{\text{def}}{=} \mathbf{e}_i \cdot \mathbf{A}_{ij}^{(r)}\, \mathbf{e}_j$, $i, j = 1, 2, 3$, are the six components of the tensor $\mathbf{A}^{(r)}$ in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. It is then clear from (1) that the maximum number of distinct components of all the tensors in the set $\{\mathbf{A}^{(1)}, \mathbf{A}^{(2)}, \ldots, \mathbf{A}^{(n)}\}$ is $6\,n-3$.

To present the methodology mentioned in Section 1, we investigate the cases $n = 1, 2, 3$, and then generalize for $n > 1$.

### 2.1 The Case $n = 1$

**Claim**: All 3 classical invariants are independent.
**Proof**: The three invariants

$$I_1^1 \stackrel{\text{def}}{=} \operatorname{tr} \mathbf{A}^{(1)} \,, \quad I_2^1 \stackrel{\text{def}}{=} \operatorname{tr} (\mathbf{A}^{(1)})^2 \,, \quad I_3^1 \stackrel{\text{def}}{=} \operatorname{tr} (\mathbf{A}^{(1)})^3 \tag{2}$$

depend on the three eigenvalues $\lambda_i$, $i = 1, 2, 3$, which are independent variables. In addition, these variables are roots of the characteristic equation having the form $\lambda^3 - J_1\,\lambda^2 + J_2\,\lambda - J_3 = 0$, where $J_1 \stackrel{\text{def}}{=} I_1^1$ , $J_2 \stackrel{\text{def}}{=} \left[(I_1^1)^2 - I_2^1\right]/2$ , $J_3 \stackrel{\text{def}}{=} \det \mathbf{A}^{(1)} = \left(I_3^1 - I_1^1\, I_2^1 + I_1^1\, J_2\right)/3$ . It is well known that this characteristic equation yields three real-valued expressions for $\lambda$ in terms of the invariants $J_i$, $i = 1, 2, 3$, and, in view of (2), in terms of the invariants $I_i^1$, $i = 1, 2, 3$. Thus, the three invariants in (2) are independent.

## 2.2 The Case $n = 2$

**Claim**: 9 of the 10 classical invariants are independent.

**Proof**: In addition to the 3 invariants in (2), the minimal integrity basis for 2 symmetric tensors, presented by Spencer [7], contains the invariants

$$I_1^2 \stackrel{\text{def}}{=} \operatorname{tr} \mathbf{A}^{(2)}\,, \quad I_2^2 \stackrel{\text{def}}{=} \operatorname{tr}(\mathbf{A}^{(2)})^2\,, \quad I_3^2 \stackrel{\text{def}}{=} \operatorname{tr}(\mathbf{A}^{(2)})^3\,, \quad I_1^{12} \stackrel{\text{def}}{=} \operatorname{tr} \mathbf{A}^{(1)}\,\mathbf{A}^{(2)},$$
$$I_2^{12} \stackrel{\text{def}}{=} \operatorname{tr}(\mathbf{A}^{(1)})^2\,\mathbf{A}^{(2)}, \quad I_3^{12} \stackrel{\text{def}}{=} \operatorname{tr} \mathbf{A}^{(1)}\,(\mathbf{A}^{(2)})^2, \quad I_4^{12} \stackrel{\text{def}}{=} \operatorname{tr}(\mathbf{A}^{(1)})^2\,(\mathbf{A}^{(2)})^2. \tag{3}$$

Let $\beta_{ij} \stackrel{\text{def}}{=} \mathbf{e}_i \cdot \mathbf{A}_{ij}^{(2)}\,\mathbf{e}_j\,, i, j = 1, 2, 3$ , where we recall from (1) that $\mathbf{e}_i$ is an eigenvector of the tensor $\mathbf{A}^{(1)}$. Clearly, the tensors $\mathbf{A}^{(1)}$ and $\mathbf{A}^{(2)}$ are uniquely determined by the nine components $\lambda_i\,, \beta_{ij},\ i, j = 1, 2, 3$, and, therefore, the ten invariants in both (2) and (3) are given in terms of these components.

In fact, it is not difficult to show that each invariant in the set $\Psi \stackrel{\text{def}}{=} \{I_1^1, I_2^1, I_3^1, I_1^2, I_2^2, I_3^2, I_1^{12}, I_2^{12}, I_3^{12}, I_4^{12}\}$ can be expressed as a polynomial of elements in the set $\Omega = \{\lambda_1, \lambda_2, \lambda_3, \beta_{11}, \beta_{22}, \beta_{33}, \beta_{12}^2, \beta_{23}^2, \beta_{13}^2, \beta_{12}\,\beta_{23}\,\beta_{13}\}$. In fact, there is a bijection between the sets $\Psi$ and $\Omega$. To show this, we only need to express the elements of $\Omega$ in terms of the elements of $\Psi$.

Thus, given that the elements of $\Psi$ are known, we obtain the elements of $\Omega$ by following the steps below.

**a)** In view of Section 2.1, the components $\lambda_1,\ \lambda_2,\ \lambda_3$ are given in terms of the invariants $I_1^1,\ I_2^1,\ I_3^1$.

**b)** The components $\beta_{11}, \beta_{22},\ \beta_{33}$ are obtained from the solution of a system of linear equations obtained from the expressions of $I_1^2, I_1^{12},\ I_2^{12}$, where $\lambda_1,\ \lambda_2,\ \lambda_3$ were determined in Step **a)**.

**c)** The terms $\beta_{12}^2, \beta_{23}^2, \beta_{13}^2$ are obtained from the solution of a system of linear equations obtained from the expressions of $I_2^2, I_3^{12},\ I_4^{12}$, where $\lambda_i,\ \beta_{ii}$, no sum on $i = 1, 2, 3$, were determined in steps **a)** and **b)**.

**d)** The term $\beta_{12}\,\beta_{23}\,\beta_{13}$ is obtained from the expression of $I_3^2$, where the other terms in this expression were determined in the previous steps.

In this way, we have shown that there is a bijection between the sets $\Psi$ and $\Omega$.

Since

$$(\beta_{12}\,\beta_{13}\,\beta_{23})^2 = \beta_{12}^2\,\beta_{13}^2\,\beta_{23}^2 \tag{4}$$

and since the integrity basis is minimal and contains the 10 invariants given by both (2) and (3), it follows from steps **a)** – **d)** above that the relation (4) yields a syzygy between the invariants. Since the elements of the set $\Omega\backslash\{\beta_{12}\,\beta_{23}\,\beta_{13}\}$ are independent, the claim is proved.

The results of this section were obtained by Rocha and Aguiar [4] and are included here for completeness of presentation. Shariff *et al.* [6] have obtained similar results by using a different approach.

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### 2.3 The Case $n = 3$

**Claim**: 15 of the 28 classical invariants are independent.

**Proof**: In addition to the 10 invariants in the expressions (2) and (3), the integrity basis for 3 symmetric tensors, presented by Spencer [7], also have the invariants

$$\begin{aligned}
&I_1^3 \overset{\text{def}}{=} \operatorname{tr} \mathbf{A}^{(3)}, \quad I_2^3 \overset{\text{def}}{=} \operatorname{tr} (\mathbf{A}^{(3)})^2, \quad I_3^3 \overset{\text{def}}{=} \operatorname{tr} (\mathbf{A}^{(3)})^3, \quad I_1^{13} \overset{\text{def}}{=} \operatorname{tr} \mathbf{A}^{(1)} \mathbf{A}^{(3)}, \\
&I_2^{13} \overset{\text{def}}{=} \operatorname{tr} (\mathbf{A}^{(1)})^2 \mathbf{A}^{(3)}, \quad I_3^{13} \overset{\text{def}}{=} \operatorname{tr} \mathbf{A}^{(1)} (\mathbf{A}^{(3)})^2, \quad I_4^{13} \overset{\text{def}}{=} \operatorname{tr} (\mathbf{A}^{(1)})^2 (\mathbf{A}^{(3)})^2, \\
&I_1^{23} \overset{\text{def}}{=} \operatorname{tr} \mathbf{A}^{(2)} \mathbf{A}^{(3)}, \quad I_2^{23} \overset{\text{def}}{=} \operatorname{tr} (\mathbf{A}^{(2)})^2 \mathbf{A}^{(3)}, \quad I_3^{23} \overset{\text{def}}{=} \operatorname{tr} \mathbf{A}^{(2)} (\mathbf{A}^{(3)})^2, \qquad (5) \\
&I_4^{23} \overset{\text{def}}{=} \operatorname{tr} (\mathbf{A}^{(2)})^2 (\mathbf{A}^{(3)})^2, \quad I_1^{123} \overset{\text{def}}{=} \operatorname{tr} \mathbf{A}^{(1)} \mathbf{A}^{(2)} \mathbf{A}^{(3)}, \quad I_2^{123} \overset{\text{def}}{=} \operatorname{tr} (\mathbf{A}^{(1)})^2 \mathbf{A}^{(2)} \mathbf{A}^{(3)}, \\
&I_3^{123} \overset{\text{def}}{=} \operatorname{tr} \mathbf{A}^{(1)} (\mathbf{A}^{(2)})^2 \mathbf{A}^{(3)}, \quad I_4^{123} \overset{\text{def}}{=} \operatorname{tr} \mathbf{A}^{(1)} \mathbf{A}^{(2)} (\mathbf{A}^{(3)})^2, \quad I_5^{123} \overset{\text{def}}{=} \operatorname{tr} (\mathbf{A}^{(1)})^2 (\mathbf{A}^{(2)})^2 \mathbf{A}^{(3)}, \\
&I_6^{123} \overset{\text{def}}{=} \operatorname{tr} (\mathbf{A}^{(1)})^2 \mathbf{A}^{(2)} (\mathbf{A}^{(3)})^2, \quad I_7^{123} \overset{\text{def}}{=} \operatorname{tr} \mathbf{A}^{(1)} (\mathbf{A}^{(2)})^2 (\mathbf{A}^{(3)})^2 .
\end{aligned}$$

Let $\gamma_{ij} \overset{\text{def}}{=} \mathbf{e}_i \cdot \mathbf{A}_{ij}^{(3)} \mathbf{e}_j \, , i, j = 1, 2, 3$ . As in Section 3.2, the tensors $\mathbf{A}^{(i)}$, $i, j = 1, 2, 3$, are uniquely determined by the 15 components $\lambda_i \, , \beta_{ij} \, , \gamma_{ij}$, $i, j = 1, 2, 3$, and, therefore, the 28 invariants in (2), (3), and (5) are given in terms of these components.

Here, the set of invariants is given by $\Psi = \{I_i^p, I_j^q, I_k^{123}\}$, $i, p = 1, 2, 3$, $j = 1, \ldots, 4$, $q = 12, 23, 13$, $k = 1, \ldots, 7$, and the set of terms in $\Omega$ will be constructed by following the steps below.

**a)** Following the steps **a)** thru **d)** in Section 3.2, it is not difficult to see that there is a bijection between the set of 17 invariants given by $\Psi_a = \{I_i^p, I_j^q\}$, $i, p = 1, 2, 3$, $j = 1, \ldots, 4$, $q = 12, 13$, and the set of 17 terms given by $\Omega_a = \{\lambda_i, \beta_{ii}, \gamma_{ii}, \beta_{ij}^2, \gamma_{ij}^2$, $\beta_{12} \, \beta_{23} \, \beta_{13}, \gamma_{12} \, \gamma_{23} \, \gamma_{13}\}$, no sum on $i, j = 1, 2, 3$, $j > i$.

Of course, not all the terms in the set $\Omega_a$ are independent. In fact, the terms $\beta_{12} \, \beta_{23} \, \beta_{13}$, $\gamma_{12} \, \gamma_{23} \, \gamma_{13}$ yield 2 relations, given by both the relation (4) and

$$(\gamma_{12} \, \gamma_{23} \, \gamma_{13})^2 = \gamma_{12}^2 \, \gamma_{23}^2 \, \gamma_{13}^2 \, . \qquad (6)$$

**b)** It is not difficult to show that each invariant in the set $\Psi_b = \{I_1^{23}, I_1^{123}, I_2^{123}\}$ can be expressed as a polynomial of elements in the set $\Omega_b = \{\beta_{12} \, \gamma_{12}, \beta_{13} \, \gamma_{13}, \beta_{23} \, \gamma_{23}\}$. In fact, there is a bijection between the sets $\Psi_b$ and $\Omega_b$.

Here, the terms in $\Omega_b$ yield the 4 relations

$$\begin{aligned}
&(\beta_{12} \, \gamma_{12})^2 = \beta_{12}^2 \, \gamma_{12}^2 \, , \quad (\beta_{13} \, \gamma_{13})^2 = \beta_{13}^2 \, \gamma_{13}^2 \, , \quad (\beta_{23} \, \gamma_{23})^2 = \beta_{23}^2 \, \gamma_{23}^2 \, , \\
&(\beta_{12} \, \gamma_{12}) \, (\beta_{23} \, \gamma_{23}) \, (\beta_{13} \, \gamma_{13}) = (\beta_{12} \, \beta_{23} \, \beta_{13}) \, (\gamma_{12} \, \gamma_{23} \, \gamma_{13}) \, , \qquad (7)
\end{aligned}$$

where $\beta_{ij}^2$, $\gamma_{ij}^2$, $i = 1, 2, 3$, $j > i$, $\beta_{12} \, \beta_{23} \, \beta_{13}$, and $\gamma_{12} \, \gamma_{23} \, \gamma_{13}$ were obtained in Step **a)**.

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**c)** It is not difficult to see that the invariants in the set $\Psi_c = \{I_2^{23}, I_3^{123}, I_5^{123}\} \cup \{I_3^{23}, I_4^{123}, I_6^{123}\}$ are given in terms of the elements in the set $\Omega_c = \{\beta_{12}\,\beta_{13}\,\gamma_{23}, \beta_{13}\,\beta_{23}\,\gamma_{12}, \beta_{23}\,\beta_{12}\,\gamma_{13}\} \cup \{\beta_{12}\,\gamma_{13}\,\gamma_{23}, \beta_{13}\,\gamma_{23}\,\gamma_{12}, \beta_{23}\,\gamma_{12}\,\gamma_{13}\}$. We have verified that, conversely, the elements in the set $\Omega_c$ can be given in terms of the invariants in the set $\Psi_c$.

Here, the terms in $\Psi_c$ yield the 13 relations

$$
\begin{aligned}
&(\beta_{12}\,\beta_{13}\,\gamma_{23})^2 = \beta_{12}^2\,\beta_{13}^2\,\gamma_{23}^2\ (*)\,, \quad (\beta_{12}\,\gamma_{13}\,\gamma_{23})^2 = \beta_{12}^2\,\gamma_{13}^2\,\gamma_{23}^2\ (*)\,,\\
&(\beta_{12}\,\beta_{13}\,\gamma_{23})\,(\beta_{13}\,\beta_{23}\,\gamma_{12})\,(\beta_{23}\,\beta_{12}\,\gamma_{13}) = \beta_{12}^2\,\beta_{13}^2\,\beta_{23}^2\,(\gamma_{12}\,\gamma_{23}\,\gamma_{13})\,,\\
&(\beta_{12}\,\gamma_{13}\,\gamma_{23})\,(\beta_{13}\,\gamma_{23}\,\gamma_{12})\,(\beta_{23}\,\gamma_{12}\,\gamma_{13}) = \gamma_{12}^2\,\gamma_{13}^2\,\gamma_{23}^2\,(\beta_{12}\,\beta_{23}\,\beta_{13})\,,\\
&(\beta_{12}\,\beta_{13}\,\gamma_{23})\,(\beta_{13}\,\beta_{23}\,\gamma_{12})\,(\beta_{23}\,\beta_{12}\,\gamma_{13}) = (\beta_{12}\,\gamma_{12})\,(\beta_{23}\,\gamma_{23})\,(\beta_{13}\ \gamma_{13})\,(\beta_{12}\,\beta_{23}\,\beta_{13})\,,\\
&(\beta_{12}\,\gamma_{13}\,\gamma_{23})\,(\beta_{13}\,\gamma_{23}\,\gamma_{12})\,(\beta_{23}\,\gamma_{12}\,\gamma_{13}) = (\beta_{12}\,\gamma_{12})\,(\beta_{23}\,\gamma_{23})\,(\beta_{13}\ \gamma_{13})\,(\gamma_{12}\,\gamma_{23}\,\gamma_{13})\,,\\
&(\beta_{12}\,\gamma_{13}\,\gamma_{23})\,(\beta_{13}\,\beta_{23}\,\gamma_{12}) = (\beta_{12}\,\beta_{23}\,\beta_{13})\,(\gamma_{12}\,\gamma_{23}\,\gamma_{13})\ \ (*)\,,
\end{aligned}
\tag{8}
$$

where $(*)$ means cyclic permutation of the indexes in the expression ($i.e., 12 \to 23 \to 31$ (or, 13) $\to 12$) and $\beta_{ij}^2$, $\gamma_{ij}^2$, $i, j = 1, 2, 3, j > i$, $\beta_{12}\,\beta_{23}\,\beta_{13}$, $\gamma_{12}\,\gamma_{23}\,\gamma_{13}$ were obtained in Step **a)** and $\beta_{12}\,\gamma_{12}, \beta_{13}\,\gamma_{13}$, $\beta_{23}\,\gamma_{23}$ were obtained in Step **b)**.

**d)** Observe from the sets $\Omega_a$, $\Omega_b$, and $\Omega_c$ that all possible combinations between the components of the tensors were considered. Thus, the two remaining invariants, $I_4^{23}$ and $I_7^{123}$, are given in terms of these combinations, which, by their turn, are given in terms of the invariants in the set $\Psi \stackrel{\text{def}}{=} \Psi_a \cup \Psi_b \cup \Psi_c$.

In summary, observe from steps **a)** – **d)** that only 15 elements in the set $\Omega \stackrel{\text{def}}{=} \Omega_a \cup \Omega_b \cup \Omega_c$ are independent. They are the elements in the set $\Omega_I \stackrel{\text{def}}{=} \Omega_a \backslash \{\beta_{12}\,\beta_{23}\,\beta_{13}, \gamma_{12}\,\gamma_{23}\,\gamma_{13}\}$. Since, by Step **a)**, there is a bijection between $\Omega_I$ and the set $\Psi_I \stackrel{\text{def}}{=} \Psi_a \backslash \{I_3^2, I_3^3\}$, we see that $\Psi_I$ contains 15 independent invariants and the claim is proved. All the other invariants depend on these invariants through either explicit expressions discussed in Step **d)** or syzygies obtained from the relations (4) and (6) thru (8).

## 2.4 The General Case of $n > 1$

**Claim**: $6\,n - 3$ of the numerous classical invariants in an $n$-minimal integrity basis are independent.

**Proof**: The components of the $n$ tensors in the set $\{\mathbf{A}^{(1)}, \mathbf{A}^{(2)}, \ldots, \mathbf{A}^{(n)}\}$, which appear in (1), yield the set $\Omega_I = \{\lambda_i, \beta_{ii}^{(r)}, (\beta_{ij}^{(r)})^2\}$, no sum on $i, j = 1, 2, 3,\ j > i,\ r = 2, \ldots, n$. The elements of this set are clearly independent and yield the $6\,n - 3$ invariants in the set $\Psi_I = \{I_1^1, I_2^1, I_3^1, I_1^r, I_2^r, I_i^{1r}\}$, $i, = 1, \ldots, 4$, $r = 2, \ldots, n$, where these invariants are defined by

$$
\begin{aligned}
&I_1^s \stackrel{\text{def}}{=} \operatorname{tr} \mathbf{A}^{(s)}\,, \quad I_2^s \stackrel{\text{def}}{=} \operatorname{tr}(\mathbf{A}^{(s)})^2\,, \quad I_3^s \stackrel{\text{def}}{=} \operatorname{tr}(\mathbf{A}^{(s)})^3\,, \quad I_1^{1r} \stackrel{\text{def}}{=} \operatorname{tr} \mathbf{A}^{(1)}\,\mathbf{A}^{(r)},\\
&I_2^{1r} \stackrel{\text{def}}{=} \operatorname{tr}(\mathbf{A}^{(1)})^2\,\mathbf{A}^{(r)}, \quad I_3^{1r} \stackrel{\text{def}}{=} \operatorname{tr} \mathbf{A}^{(1)}\,(\mathbf{A}^{(r)})^2, \quad I_4^{1r} \stackrel{\text{def}}{=} \operatorname{tr}(\mathbf{A}^{(1)})^2\,(\mathbf{A}^{(r)})^2,
\end{aligned}
\tag{9}
$$

 

where $s = 1, 2, \ldots, n$. Following the arguments of Step **a)** in Case $n = 3$, it is not difficult to see that there is a bijection between the sets $\Omega_I$ and $\Psi_I$ and, therefore, that the invariants of $\Psi_I$ are independent, establishing the proof of the claim.

These independent invariants can now be used to establish $n - 1$ syzygies from the relations

$$\left(\beta_{12}^{(r)}\,\beta_{23}^{(r)}\,\beta_{13}^{(r)}\right)^2 = \left(\beta_{12}^{(r)}\right)^2\,\left(\beta_{23}^{(r)}\right)^2\,\left(\beta_{13}^{(r)}\right)^2, \qquad r = 2, \ldots, n, \tag{10}$$

where the term $\beta_{12}^{(r)}\,\beta_{23}^{(r)}\,\beta_{13}^{(r)}$ is a polynomial of the invariants in the set $\Psi_I \cup \{I_3^r\}$, $r = 2, \ldots, n$.

Next, following the arguments of Step **b)** in Case $n = 3$, we see that there are bijections between the sets $\{I_1^{rs}, I_1^{1rs}, I_2^{1rs}\}$ and $\{\beta_{12}^{(r)}\,\beta_{12}^{(s)}, \beta_{13}^{(r)}\,\beta_{13}^{(s)}, \beta_{23}^{(r)}\,\beta_{23}^{(s)}\}$ for $r, s = 2, \ldots, n$, $s > r$, which allow to establish $2\,(n-1)\,(n-2)$ relations having the forms given in (7).

Similarly, following the arguments of Step **c)** in Case $n = 3$, we see that there are bijections between the sets $\{I_2^{rs}, I_3^{1rs}, I_5^{1rs}\} \cup \{I_3^{rs}, I_4^{1rs}, I_6^{1rs}\}$ and $\{\beta_{12}^{(r)}\,\beta_{13}^{(r)}\,\beta_{23}^{(s)}, \beta_{13}^{(r)}\,\beta_{23}^{(r)}\,\beta_{12}^{(s)}, \beta_{23}^{(r)}\,\beta_{12}^{(r)}\,\beta_{13}^{(s)}\} \cup \{\beta_{12}^{(r)}\,\beta_{13}^{(s)}\,\beta_{23}^{(s)}, \beta_{13}^{(r)}\,\beta_{23}^{(s)}\,\beta_{12}^{(s)}, \beta_{23}^{(r)}\,\beta_{12}^{(s)}\,\beta_{13}^{(s)}\}$ for $r, s = 2, \ldots, n$, $s > r$, which allow to establish $13\,(n-1)\,(n-2)/2$ relations having the forms given in (8).

Similarly to Step **d)** in Case $n = 3$, the invariants $I_4^{rs}$ and $I_7^{1rs}$, given by (9.g) and an expression similar to the last definition in (5), are expressed as combinations of the terms introduced above, which, by their turn, are given in terms of the invariants also introduced above.

Since the aim of this work is not to present a complete analysis of all possible relations between the invariants, we leave this task for future work and, in the next section, present conclusions that generalize the observations of Case $n = 3$.

## 3 Conclusions

The minimal integrity basis presented by Spencer [7] has the $6\,n - 3$ independent invariants in the set $\Psi_I$ presented in Section 2.4. All the other invariants depend on these invariants through either implicit or explicit relations, as discussed below.

Provided that all the eigenvalues of $\mathbf{A}^{(1)}$ are distinct from each other, we can find non-polynomial expressions for some of the invariants in terms of the remaining ones. We then find a set having less invariants than the invariants in the minimal integrity basis. The number of invariants in this set has only been determined for Case $n = 3$ and is equal to 26.

The methodology introduced in this work yields relations between the components of the tensors $\mathbf{A}^{(r)}$, $r = 1, \ldots, n$, which can be used to find syzygies for the invariants. For Case $n = 3$, we have shown that these relations together with the 2 non-polynomial expressions referred to above yield 21 relations.

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