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Three-parameter Mittag-Leffler function with an integral representation on the positive real axis

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Abstract. In this paper, we use the methodology for evaluation of the inverse Laplace transform, proposed by M. N. Berberan-Santos, to show that the three-parameter Mittag-Leffler function has an integral representation on the positive real axis. Some of integrals are also presented.

Keywords. Inverse Laplace transform, Mittag-Leffler functions, Integral representation, Fractional calculus.

Introduction

The Mittag-Leffler function, introduced in 1902 by M. G. MittagLeffler [16], is important in many fields and some applications can be found, for example, in: description of the anomalous dielectric properties, probability theory, statistics, viscoelasticity, random walks and dynamical systems [6–8, 11, 14, 18, 19]. Successively, generalizations of Mittag-Leffler function were proposed [20]. These functions play fundamental role in arbitrary order calculus, popularly known as fractional calculus [10,13,15], as well as the exponential function play in integer order calculus.

The classical Laplace transform is one of the most widely tools used in the literature for solving integral equations and ordinary or partial differential equations, involving integer or fractional order derivatives [1, 5, 21, 24]. It is also used in many others applications such as electrical circuit solving and signal processing [12, 23]. In general, the Laplace inversion is done numerically due to the impossibility of the exact inversion by means of an integration on the complex plane [4].

M. N. Berberan-Santos [2] proposed a new methodology for evaluation of the numerical inverse Laplace transform, without using integration on the complex plane, which was published in 2005, and its methodology was used recently, for instance, to discuss the

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luminescence decay of inorganic solids [22], and to obtain an integral representation of Mittag-Leffler relaxation function, a special one-parameter Mittag-Leffler function [3].

In this paper, with the methodology of inversion of the Laplace transform proposed by M. N. Berberan-Santos, we express the three-parameter Mittag-Leffler function as an integral on the positive real axis. The paper is organized as follows: in Section 1, we present the methodology of inversion of the Laplace transform and some preliminaries concepts. In Section 2, using this methodology, we express the integral representation of three-parameter Mittag-Leffler function and we use the results from this study to discuss, in Section 3, a class of improper integrals, expressing them in terms of the Mittag-Leffler functions. Concluding remarks close the paper.

1 Preliminaries

The three-parameter Mittag-Leffler function, introduced by Prabhakar [20], is defined for complex $z \in \mathbb{C}$, $\alpha, \beta, \gamma \in \mathbb{C}$, $\mathcal{R}(\alpha) > 0$, $\mathcal{R}(\beta) > 0$ and $\mathcal{R}(\gamma) > 0$ by[4]:

$$\mathrm{E}_{\alpha,\beta}^{\gamma}(z) = \sum_{j=0}^{\infty} \frac{(\gamma)_j \, z^j}{\Gamma(\alpha j + \beta)\, j!}. \tag{1}$$

Taking $\gamma = 1$ in equation (1), we get the two-parameter Mittag-Leffler function:

$$\mathrm{E}_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + \beta)}. \tag{2}$$

If $\beta = 1$ in equation (2), we get the classical Mittag-Leffler function [17, 25]:

$$\mathrm{E}_{\alpha}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + 1)}. \tag{3}$$

Let $f(t)$ be a real function of (time) variable $t \geq 0$. The Laplace transform of f, denoted by $\mathrm{F}(s) = \mathscr{L}[f](s)$, is defined as follows:

$$\mathscr{L}[f](s) = \mathrm{F}(s) = \int_0^{\infty} e^{-st} f(t) \mathrm{d}t, \tag{4}$$

whenever the integral converges, where $s = \sigma + i\tau$, with σ and τ real numbers, and $\mathrm{F}(s) = 0$ for $\sigma < 0$. The expression for evaluation of the inverse Laplace transform, proposed by M. N. Berberan-Santos [2], is given by[5]

$$f(t) = \frac{e^{\sigma t}}{\pi} \int_0^{\infty} \left[\mathcal{R}\left[\mathrm{F}(\sigma + i\tau)\right] \cos(t\tau) - \mathcal{I}\left[\mathrm{F}(\sigma + i\tau)\right] \sin(t\tau)\right] \mathrm{d}\tau, \tag{5}$$

for any real number σ satisfying the condition $\sigma > \sigma_0 > 0$, and $t > 0$. The expression in equation (5) recovers the real function whose Laplace transform is known.

[4] $(\gamma)_j := \dfrac{\Gamma(\gamma + j)}{\Gamma(\gamma)}$ is the Pochhammer symbol and $\Gamma(\alpha) = \displaystyle\int_0^{\infty} t^{\alpha-1} e^{-t} \mathrm{d}t$ is the Gamma function.

[5] $\mathcal{R}[s]$ indicates the real part of s and the imaginary part is denoted by $\mathcal{I}[s]$.

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Remark 1. Writing $s = \sigma + i\tau$ in equation (4) and being $f(t) = 0$ for $t < 0$, we can write

$$\mathrm{F}(s) = \mathrm{F}(\sigma + i\tau) = \int_{-\infty}^{\infty} e^{-\sigma t} f(t) e^{-i\tau t} \mathrm{d}t. \tag{6}$$

Equation (6) represents the Fourier transform of the function $\varphi(t) = e^{-\sigma t} f(t)$. Then, with the conditions imposed on f, the function $\mathrm{F}(s)$ converges absolutely for $\mathcal{R}[s] = \sigma > \sigma_0 > 0$. This implies that $\varphi(t) = e^{-\sigma t} f(t)$ is absolutely integrable, and we may evaluate the inverse Fourier transform of $\varphi(t) = e^{-\sigma t} f(t)$ [see J. L. Schiff ([21], p. 151)].

2 Main result

Consider the three-parameter Mittag-Leffler type function $f(t) = t^{\beta-1}\mathrm{E}^{\gamma}_{\alpha,\beta}(\lambda t^{\alpha})$, over a restricted domain $\Omega = [0, \infty)$. Suppose also that $\alpha > 0$, $\beta > 0$ and $\gamma > 0$ in equation (1). For $t \in \Omega$ and λ a real number, the Laplace transform of $f(t)$ is given by

$$\mathscr{L}\left[t^{\beta-1}\mathrm{E}^{\gamma}_{\alpha,\beta}(\lambda t^{\alpha})\right](s) = \frac{s^{\alpha\gamma-\beta}}{(s^{\alpha} - \lambda)^{\gamma}}, \qquad \text{for } |\lambda s^{-\alpha}| < 1. \tag{7}$$

Theorem 1. Let $\alpha > 0$, $\beta > 0$, $\gamma > 0$ and $\lambda \in \mathbb{R}$. The three-parameter Mittag-Leffler function has the following integral representation on the positive real axis

$$\mathrm{E}^{\gamma}_{\alpha,\beta}(\lambda t^{\alpha}) = \frac{t^{1-\beta} e^{\sigma t}}{\pi} \int_0^{\infty} \frac{r^{\alpha\gamma-\beta}}{\tilde{r}} \cos\left[\theta(\alpha\gamma - \beta) - \tilde{\theta} + t\tau\right] \mathrm{d}\tau, \tag{8}$$

for $t > 0$, where $\sigma > \sigma_0$ and σ_0, r, θ, $\tilde{\theta}$ and $\tilde{r}$ are defined by equations:

$$\sigma_0 = |\lambda|^{\frac{1}{\alpha}}. \tag{9}$$

$$r = r(\sigma, \tau) = \sqrt{\sigma^2 + \tau^2}. \tag{10}$$

$$\theta = \theta(\sigma, \tau) = \arccos\left(\frac{\sigma}{\sqrt{\sigma^2 + \tau^2}}\right) = \arcsin\left(\frac{\tau}{\sqrt{\sigma^2 + \tau^2}}\right). \tag{11}$$

$$\tilde{r}^{\frac{1}{\gamma}} \cos\left(\frac{\tilde{\theta}}{\gamma}\right) = r^{\alpha}\cos(\theta\alpha) - \lambda \qquad \text{and} \qquad \tilde{r}^{\frac{1}{\gamma}} \sin\left(\frac{\tilde{\theta}}{\gamma}\right) = r^{\alpha}\sin(\theta\alpha). \tag{12}$$

Proof. The function $\mathrm{F}(s) = \dfrac{s^{\alpha\gamma-\beta}}{(s^{\alpha} - \lambda)^{\gamma}}$ is the Laplace transform of $f(t) = t^{\beta-1}\mathrm{E}^{\gamma}_{\alpha,\beta}(\lambda t^{\alpha})$. The complex parameter s can be written as $s = \sigma + i\tau = re^{i\theta}$. In this way, we get equations (10) and (11). Rewriting the expression $(s^{\alpha} - \lambda)^{\gamma}$ in the denominator of $\mathrm{F}(s)$ in the form $(s^{\alpha} - \lambda)^{\gamma} = \tilde{r}e^{i\tilde{\theta}}$ and using equations (10) and (11) in the left-side, we get equation (12). Through manipulation of $\mathrm{F}(s)$ we can separate its real and imaginary parts and use the equation (5) to find integral representation in equation (8). If we choose $\sigma > \sigma_0 = |\lambda|^{\frac{1}{\alpha}}$, then the inequality $|\lambda s^{-\alpha}| < 1$ is satisfied. ∎

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3 Improper integrals

In what follows we will discuss particular examples obtained for specific values of the parameters appearing in equation (8).

Example 1. Let $f(t) = \mathrm{E}_{1,\beta}^{\beta}(t)$. We consider $\alpha = 1$, $\beta = \gamma$ and $\lambda = 1$ in equation (8). We have $\alpha\gamma - \beta = 0$ and from equation (9), we can choose $\sigma = 2$, we get

$$r\,\cos(\theta) = 2 \qquad \text{and} \qquad r\,\sin(\theta) = \tau. \tag{13}$$

Then, by means of equations (12) and (13), we obtain

$$\tilde{\theta} = \beta\,\arctan(\tau) \qquad \text{and} \qquad \tilde{r} = \left(1+\tau^2\right)^{\frac{\beta}{2}}. \tag{14}$$

Substituting equations (13) and (14) into equation (8), we can write an interesting integral representation for a particular three-parameter Mittag-Leffler function which generalizes some known results:

$$\mathrm{E}_{1,\beta}^{\beta}(t) = \frac{t^{1-\beta}\,e^{2t}}{\pi}\int_0^\infty \frac{\cos\left[\beta\,\arctan(\tau) - t\tau\right]}{(1+\tau^2)^{\frac{\beta}{2}}}\,\mathrm{d}\tau, \tag{15}$$

for $t > 0$ and $\beta > 0$. Taking $\beta = 1$ in equation (15), we have an integral representation for the exponential function[6] for $t > 0$:

$$e^{-t} = \frac{1}{\pi}\int_0^\infty \frac{\cos(t\tau) + \tau\,\sin(t\tau)}{1+\tau^2}\,\mathrm{d}\tau. \tag{16}$$

Example 2. Let $f(t) = {}_1\mathrm{F}_1(\gamma;\beta;t)$ be a confluent hypergeometric function [13]. It can be expressed in terms of the Mittag-Leffler function: $\mathrm{E}_{1,\beta}^{\gamma}(t) = \dfrac{1}{\Gamma(\beta)}\,{}_1\mathrm{F}_1(\gamma;\beta;t)$. Using equation (8) we can obtain an integral representation for this confluent hypergeometric function. Indeed, taking $\lambda = 1$ and $\alpha = 1$ in equation (8) and choosing $\sigma = 2$ from equation (9) we get

$$\mathrm{E}_{1,\beta}^{\gamma}(t) = \frac{t^{1-\beta}\,e^{2\,t}}{\pi}\int_0^\infty \frac{r^{\gamma-\beta}}{\tilde{r}}\,\cos\left[\theta\,(\gamma-\beta) - \tilde{\theta} + t\tau\right]\mathrm{d}\tau,$$

or in a different form,

$${}_1\mathrm{F}_1(\gamma;\beta;t) = \frac{\Gamma(\beta)\,t^{1-\beta}\,e^{2\,t}}{\pi}\int_0^\infty \frac{r^{\gamma-\beta}}{\tilde{r}}\,\cos\left[\theta\,(\gamma-\beta) - \tilde{\theta} + t\tau\right]\mathrm{d}\tau. \tag{17}$$

As $\sigma = 2$, from equations (10) and (11), we have

$$r = \sqrt{4+\tau^2} \qquad \text{and} \qquad \theta = \arccos\left(\frac{2}{\sqrt{4+\tau^2}}\right). \tag{18}$$

Using equation (12) we obtain

$$\tilde{\theta} = \gamma\,\arctan(\tau) \qquad \text{and} \qquad \tilde{r} = \left(1+\tau^2\right)^{\gamma/2}. \tag{19}$$

[6]In this regard see also S. Gradshteyn [9]. The sum of equations 1 and 2 in [9], p. 424, with $\beta = 1$ and $a = t$, results in equation (16).

Finally, substituting equations (18) and (19) into equation (17), we obtain the integral representation for the confluent hypergeometric function:

$$ {}_1\mathrm{F}_1\left(\gamma;\beta;t\right) = \frac{\Gamma(\beta)\, t^{1-\beta}\, e^{2\,t}}{\pi} \int_0^\infty \phi\left(t,\tau\right)\, \mathrm{d}\tau, \tag{20} $$

for $t > 0$, where $\gamma > 0$, $\beta > 0$, and

$$ \phi\left(t,\tau\right) = \frac{\left(4+\tau^2\right)^{(\gamma-\beta)/2}}{\left(1+\tau^2\right)^{\gamma/2}} \cos\left[(\gamma-\beta)\arccos\left(\frac{2}{\sqrt{4+\tau^2}}\right) - \gamma\,\arctan\left(\tau\right) + t\tau\right]. \tag{21} $$

Example 3. Let $f(t) = \mathrm{E}_{\alpha,\alpha}(-t^\alpha)$. We consider $\alpha = \beta$, $\gamma = 1$ and $\lambda = -1$ in equation (8). In this case, $\alpha\gamma - \beta = 0$ and we will choose $\sigma = 2$ from equation (9). We have

$$ \mathrm{E}_{\alpha,\alpha}(-t^\alpha) = \frac{t^{1-\alpha}\, e^{2t}}{\pi} \int_0^\infty \frac{\cos(t\tau)\cos\tilde{\theta} + \sin(t\tau)\sin\tilde{\theta}}{\tilde{r}}\, \mathrm{d}\tau, \tag{22} $$

where $t > 0$ and $\tilde{\theta}$ and $\tilde{r}$ are given by equation (12). In particular, taking $\alpha = 2$ in equation (22), as $\sigma = 2$, it follows that

$$ r\cos\theta = 2 \qquad \text{and} \qquad r\sin\theta = \tau. \tag{23} $$

Substituting $\alpha = 2$ and $\lambda = -1$ into equation (12) and using equation (23), we have

$$ \tilde{r}\cos\tilde{\theta} = r^2\cos(2\theta) + 1 = 5 - \tau^2 \qquad \text{and} \qquad \tilde{r}\sin\tilde{\theta} = r^2\sin(2\theta) = 4\tau. \tag{24} $$

Multiplying the integrand in equation (22) by $\dfrac{\tilde{r}}{\tilde{r}}$, and substituting equations (23) and (24) into (22), we thus derive the following assertion:

$$ \mathrm{E}_{\alpha,\alpha}(-t^\alpha) = \frac{t^{1-\alpha}\, e^{2t}}{\pi} \int_0^\infty \frac{(5-\tau^2)\cos(t\tau) + 4\tau\,\sin(t\tau)}{\tau^4 + 6\tau^2 + 25}\, \mathrm{d}\tau. \tag{25} $$

If $\alpha = 2$ in equation (25), we have

$$ e^{-2t}\,\sin\,t = \frac{1}{\pi} \int_0^\infty \frac{(5-\tau^2)\cos(t\tau) + 4\tau\,\sin(t\tau)}{\tau^4 + 6\tau^2 + 25}\, \mathrm{d}\tau, \tag{26} $$

because $\mathrm{E}_{2,2}(z) = \dfrac{\sinh\sqrt{z}}{\sqrt{z}}$ imply $\mathrm{E}_{2,2}\left(-t^2\right) = \dfrac{\sinh it}{it} = \dfrac{\sin t}{t}$.

4 Concluding remarks

We build an integral representation for the three-parameter Mittag-Leffler function on the positive real axis using the inversion of the Laplace transform, without contour integration, proposed by M. N. Berberan-Santos. This representation can express convergent improper integrals in terms of trigonometric functions by means of the Mittag-Leffler functions and the presented examples complement corresponding integral representations.

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