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Pólya's Looking Back: an Example with Leibniz's Integrating Factor and Linear Difference Equations

Luís Henrique de Santana[1]
Departamento de Matemática, UDESC, Joinville, SC
José Rafael Santos Furlanetto[2]
Departamento de Matemática, UDESC, Joinville, SC

Abstract. Polya's four phases have been well-known worldwide and have improved our problem solving skills through the years. However, it has been hard for us to find good examples of solutions obtained by using Polya's recommendations that could be useful in our courses, we start to develop some by ourselves. In special, the looking back seems to be most neglected one among all four phases. This article shows how looking back, *a la* Pólya, at the classical Leibniz's integrating factor lead us to an integrating factor method for first order linear difference equations.

Keywords. Polya's Looking Back, Problem Solving, Differential Equation, Difference Equation, Integrating Factor

1 Introduction

Most introductory books on ordinary differential equations – for instance, [1] – present the integrating factor method for solving first order linear differential equations

$$y'(t) + p(t)\,y(t) = q(t)\,, \tag{1}$$

where p and g are given continuous functions. We owe this method to Leibniz and its first step involves multiplying equation (1) by a certain function $\mu(t)$, thus

$$\mu(t)\,y'(t) + p(t)\,\mu(t)\,y(t) = \mu(t)\,q(t)\,. \tag{2}$$

The second step is to pick a function $\mu(t)$ such that $\mu'(t) = p(t)\,\mu(t)$, for instance, $\mu(t) = e^{\int_{t_0}^{t} p(\tau)\,d\tau}$. Therefore, equation (2) becomes

$$(\mu(t)\,y(t))' = \mu(t)\,g(t)\,. \tag{3}$$

[1]desantanalh@gmail.com
[2]jose.furlanetto@udesc.br

Then, by integration we obtain

$$\mu(t)\, y(t) = \int_{t_0}^{t} \mu(\tau)\, g(\tau) d\tau + k\,,$$

from which it follows that the general solution of Equation (1) is

$$y(t) = \frac{1}{\mu(t)} \left(\int_{t_0}^{t} \mu(\tau)\, g(\tau) d\tau + k \right)\,, \tag{4}$$

where k is an arbitrary constant.

It is shown in [2] that the solution of an difference equation

$$\begin{cases} x[n+1] = x[n] + b[n]\,, \\ x[0] = x_0\,, \end{cases} \tag{5}$$

where $n \in \mathbb{Z}_+$, is given by

$$x[n] = x_0 + \sum_{i=0}^{n-1} b[i]\,. \tag{6}$$

Moreover, Elaydi also solved in [2] by the method of finite induction the non-homogeneous difference equation

$$\begin{cases} y[n+1] = a[n]y[n] + g[n]\,, \\ y[0] = y_0\,, \end{cases} \tag{7}$$

where $a[n] \neq 0$ for each $n \in \mathbb{Z}_+$.

On his best-selling [4], Pólya separates the problem solving process into four phases, namely, *understanding the problem*, *devising a plan*, *carrying out the plan* and *looking back*. The influence of Polya's four phases on mathematical education can be noted, for instance, by the fact that almost every article submitted to the 1980 yearbook of the National Council of Teachers of Mathematics, which were concerned on problem-solving, references to them [3]. Polya's four phases have been well-known worldwide and have improved our problem solving skills through the years. However, it has been hard for us to find good examples of solutions obtained by using Polya's recommendations that could be useful in our courses, we start to develop some by ourselves. In special, the looking back phase seems to be most neglected one among all four phases. It might be due the fact that many well-done solutions obtained by using Pólya's approach, for some reason, were never even submitted for publishing by its authors.

This article focus on the last phase, the looking back, whose suggestions are: *can you check the result? can you check the argument?*; *can you derive the result differently? can you see it at a glance?*; *can we use the result, or the method, for some other problem?*. Notice that one could easily check that y given by (4) is a solution of equation (1). Furthermore, it is not hard to check the arguments in each step of Leibniz's integrating factor method too. There is another classical way for solving equation (1): the variation of parameters method. However, can we use the Leibniz's integrating factor method for some other problem? Since difference equations are the discrete-time analogous of differential equations,

Question 1.1. *Can we use a sort of Leibniz's integrating factor method for equation (7)?*

That was our hope and our starting point.

Here we intend to to describe, as good as we could, how looking back at Leibniz's integrating factor lead us to an integrating factor method for the difference equation (7). In the next section, we present the answer for question 1.1. Although the authors could not find this solution presented here elsewhere, they are not quite sure that it is a new one. The section 3 presents the solution and two new questions.

2 First Order Linear Difference Equation

Breaking the Leibniz's integrating factor down, we could perceive that it reduces the problem (1) into the easier problem (2). So, we start trying to find a way out to reduce equation (7) to an easier difference equation. Eventually we have gessed that the easier equation could be (5).

Inspired in the integrating method, let us multiply the first equation of problem (7) by $\mu[n+1]$, *i.e.*,

$$\mu[n+1]y[n+1] - \mu[n+1]a[n]y[n] = \mu[n+1]g[n] \,. \tag{8}$$

If this function μ is such that

$$\mu[n] = a[n]\mu[n+1] \,, \tag{9}$$

we have

$$\mu[n+1]y[n+1] - \mu[n]y[n] = \mu[n+1]g[n] \,. \tag{10}$$

If we denote

$$z[n] = \mu[n]y[n] \,, \tag{11}$$

where $n \in \mathbb{Z}_+$, and

$$z[0] = \mu[0]y[0] \,, \tag{12}$$

problem (7) is reduced to

$$\begin{cases} z[n+1] - z[n] = h[n] \,, \\ z[0] = z_0 \,, \end{cases} \tag{13}$$

where

$$h[n] = \mu[n+1]g[n] \,. \tag{14}$$

We already know that the solution of problem (13) is given by

$$z[n] = z_0 + \sum_{r=0}^{n-1} h[r] \,. \tag{15}$$

Therefore, it is sufficient to prove that there is such μ satisfying Equation (9). This equation can be rewritten as

$$\mu[n+1] = (a[n])^{-1}\mu[n] \,. \tag{16}$$

It is easy to see that

$$\mu[0] = 1 \quad \text{and} \quad \mu[n] = \left(\prod_{i=0}^{n-1} (a[i])^{-1}\right) \tag{17}$$

is a solution of equation (16). From equations (11), (12) and (17), we have

$$z_0 = y_0 \quad \text{and} \quad z[n] = y[n] \left(\prod_{i=0}^{n-1} (a[i])^{-1}\right). \tag{18}$$

From Equations (15) and (18),

$$y[n] = y_0 \left(\prod_{i=0}^{n-1} a[i]\right) + \sum_{r=0}^{n-1} \left(\prod_{i=0}^{n-1} a[i]\right) h[r] \,.$$

Thus, from Equation (14),

$$y[n] = y_0 \left(\prod_{i=0}^{n-1} a[i]\right) + \sum_{r=0}^{n-1} \left(\prod_{i=0}^{n-1} a[i]\right) \mu[r+1] g[r]$$

and, at last, from equation (17),

$$y[n] = y_0 \left(\prod_{i=0}^{n-1} a[i]\right) + \sum_{r=0}^{n-1} \left(\prod_{i=0}^{n-1} a[i]\right) \left(\prod_{k=0}^{r} (a[k])^{-1}\right) g[r]$$

$$y[n] = y_0 \left(\prod_{i=0}^{n-1} a[i]\right) + \sum_{r=0}^{n-1} \left(\prod_{i=r+1}^{n-1} a[i]\right) g[r] \,.$$

Therefore, question 1.1 is answered.

3 Conclusion

This solution of question 1.1 could be presented as an example of Pólya's looking back phase either in a differential equation or in a difference equation course. Furthermore, once we have presented, in [5], an integrating factor method for second order differential equations with constant coefficients,

Exercise 3.1. *Can we use the method of question 1.1 and the method of [5] for solving*

$$y[n+2] + b\, y[n+1] + c\, y[n] = g[n] \,, \quad y[0] = y_0 \,,$$

where b and c are given constants, and $n \in \mathbb{Z}_+$?

Moreover,

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Exercise 3.2. *Can we use either the method or the result of question 1.1 for solving*

$$\begin{cases} y[n+1] = a_{1,1}[n]\, y[n] + a_{1,2}[n]\, w[n] + g_1[n]\,, \\ w[n+1] = a_{2,1}[n]\, y[n] + a_{2,2}[n]\, w[n] + g_2[n]\,, \end{cases} \tag{19}$$

where $n \in \mathbb{Z}_+$*?*

We have been practicing Polya's advices with our students in the following manner:

- at first, we spend some time proposing some problems and asking the students to follow bit-by-bit all the four phases; we believe it gives the students plenty opportunity to memorize and to become acquainted with most of Pólya's suggestions;
- later, we present some solved problem as the start point for an looking phase; we believe that the this stage resembles more the mathematical research activity.

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