Proceeding Series of the Brazilian Society of Applied and Computational Mathematics, v. 6, n. 1, 2018.

Trabalho apresentado no XXXVII CNMAC, S.J. dos Campos - SP, 2017.

Proceeding Series of the Brazilian Society of Computational and Applied Mathematics

Fritz-John Necessary Condition for Optimization Problem with an Interval-valued Objective Function

Ulcilea A. Severino Leal[1]
Campus Universitário de Iturama, UFTM, Iturama, MG
Geraldo Nunes Silva[2]
Department of Applied Mathematics, UNESP/IBILCE, São José do Rio Preto, SP
Weldon A. Lodwick[3]
Department of Mathematics and Statistical Sciences,UCDenver, Denver, CO

Abstract. This paper proposes Fritz-John necessary condition for optimization problem with interval-valued objective function. For this, we consider the LU concept of order relation in the interval space and the concrpt of E-differentiable for interval-valued functions.

Keywords. interval-valued function, E-differentiable, necessary condition

1 Introduction

Theory and methods of mathematical programming are important components of applied mathematics. Due to a necessity to broaden these theories to encompass more applications, many mathematicians have focused on optimization. With their advances in the methods and theoretical results, the field of optimization has achieved significant results in many application areas.

In most cases, the coefficients of optimization problems are considered to be deterministic and have fixed valued. However, there are many situations where this consideration is not valid. In stochastic programming problems and fuzzy programming problems, the coefficients are viewed as random variables and fuzzy sets, respectively. Thus, the probability distributions and membership functions for theses two classes of problems need to be known. Therefore, interval-valued optimization problems may provide an alternative theory for considering the uncertainty of these problems.

Wu [12] obtained the KKT conditions in an optimization problem with an interval-valued objective function using the concept of a weakly differentiable interval-valued function called the H-derivative. Chalco-Cano et al. [3] obtained KKT conditions using the gH-derivative and compared these with previous results to show certain advantages of their approach.

In this study, we obtain the Fritz-John necessary condition using the concepts of E-diferentiable for interval-valued functions.

[1]ulcilea.leal@uftm.edu.br
[2]gsilva@ibilce.unesp.br
[3]Weldon.Lodwick@ucdenver.edu

2 The extremal derivative

This section describes the basic elements of derivative. If $n = 1$, $\mathbb{I}(\mathbb{R})$ denotes the set of closed and bounded intervals of the real line, i.e., $\mathbb{I}(\mathbb{R}) = \{[\underline{a}, \overline{a}]; \underline{a}, \overline{a} \in \mathbb{R} \text{ and } \underline{a} \leq \overline{a}\}$, where $\underline{a}$ and $\overline{a}$ are the lower and upper bounds of an interval $A \in \mathbb{I}(\mathbb{R})$, respectively. An interval-valued function $F : [t_0, t_1] \to \mathbb{I}(\mathbb{R})$ is denoted by $F(t) = [\underline{f}(t), \overline{f}(t)]$ such that $\underline{f}(t) \leq \overline{f}(t)$ for all $t \in [t_0, t_1]$. The functions $\underline{f}$ and $\overline{f}$ are now called, respectively, the lower and upper functions of F.

Definition 2.1. *An interval-valued function $F : X \in \mathbb{R} \to \mathbb{I}(\mathbb{R})$ is endpoint differentiable (E-differentiable, for short) at $\boldsymbol{x}_0 \in X$ if and only if the real-valued functions $\underline{f}$ and $\overline{f}$ are differentiable at $\boldsymbol{x}_0$. Furthermore, F is continuously E-differentiable at $\boldsymbol{x}_0$ if all the partial derivatives of the extremal functions exist in some neighborhoods of $\boldsymbol{x}_0$ and are continuous at $\boldsymbol{x}_0$.*

3 Interval-valued optimization

This section present the optimization problem with interval-valued objective function. Let X the feasible set, given that $X = \{x \in \mathbb{R}^n | \ g_i(x) \leq 0, \ \ i = 1, ..., m\}$, where $g : \mathbb{R}^n \to \mathbb{R}^m$ is a real vector-valued function, and $F : \mathbb{R}^n \to \mathbb{I}(\mathbb{R})$ is an interval-valued objective function. That is, we consider the following problem:

$$\begin{aligned} &\min \quad F(\mathbf{x}) = [\underline{f}(\mathbf{x}), \overline{f}(\mathbf{x})] \\ &\text{subject to:} \\ &g_i(\mathbf{x}) \leq 0, \ \ i = 1, ..., m. \end{aligned} \tag{1}$$

The first thing to be considered is how the solutions of problem (1) are "ranked" since we have an interval-valued objective function. The solution concept for this problem is defined from a partial order relation in the space $\mathbb{I}(\mathbb{R})$. This is similar to the solution concept of Pareto optimal solutions used in multi-objective programming problems.

We study the problem (1) using the usual order relation $\leq_{LU}$ in $\mathbb{I}(\mathbb{R})$ as follows.

Let $A = [\underline{a}, \overline{a}]$ and $B = [\underline{b}, \overline{b}]$ be two intervals. The order relation $\leq_{LU}$ is defined by

$$A \leq_{LU} B \text{ if and only if } \underline{a} \leq \underline{b} \text{ and } \overline{a} \leq \overline{b}.$$

For this order relation, the solution concept for the optimization problem is defined by (see [3]): let $\widehat{\mathbf{x}}$ be a feasible solution of problem (1), i.e., $\widehat{\mathbf{x}} \in X$. Then, $\widehat{\mathbf{x}}$ is a LU-solution of problem (1), if there exists no $\mathbf{x} \in X$ such that $F(\mathbf{x}) <_{LU} F(\widehat{\mathbf{x}})$.

Theorem 3.1. *If $\widehat{\boldsymbol{x}}$ is a LU-solution of the problem (1), then $\widehat{\boldsymbol{x}}$ solves the following two problems:*

$$(P_1) \begin{cases} \min \quad \underline{f}(\boldsymbol{x}) \\ \textit{subject to:} \\ \overline{f}(\boldsymbol{x}) \leq \overline{f}(\widehat{\boldsymbol{x}}) \\ g_i(\boldsymbol{x}) \leq 0, \ \ i = 1, ..., m, \end{cases} \quad \textit{and} \quad (P_2) \begin{cases} \min \quad \overline{f}(\boldsymbol{x}) \\ \textit{subject to:} \\ \underline{f}(\boldsymbol{x}) \leq \underline{f}(\widehat{\boldsymbol{x}}) \\ g_i(\boldsymbol{x}) \leq 0, \ \ i = 1, ..., m. \end{cases}$$

DOI: 10.5540/03.2018.006.01.0369

Reciprocally, if $\widehat{\boldsymbol{x}}$ *solves* (P_1) *and* (P_2)*, then* $\widehat{\boldsymbol{x}}$ *is a LU-solution of the problem (1).*

We need the concept of convexity for interval-valued functions before presenting the result. According Chalco-Cano et al. [3], [8] and [14] let F be an interval-valued function defined on a convex set $X \subset \mathbb{R}^n$. Then,

1. F is LU-convex at $\mathbf{x}^*$ if

$$F(\lambda \mathbf{x}^* + (1-\lambda)\mathbf{x}) \leq_{LU} \lambda F(\mathbf{x}^*) + (1-\lambda)F(\mathbf{x})$$

for all $\lambda \in (0,1)$ and each $\mathbf{x} \in X$.

Similarly,

1. F is LU-convex at x_i^* if

$$\begin{aligned}&F(x_1, ...x_{i-1}, \lambda x_i^* + (1-\lambda)x_i, x_{i+1}, ..., x_n) \leq_{LU} \\ &\lambda F(x_1, ...x_{i-1}, x_i^*, x_{i+1}, ..., x_n) + (1-\lambda)F(x_1, ...x_{i-1}, x_i, x_{i+1}, ..., x_n)),\end{aligned}$$

for all $\lambda \in (0,1)$ and each x_i.

Furthermore, Chalco-Cano et al. [3], [8] and [14] proved that: let X be a convex subset of $\mathbb{R}^n$ and F be an interval-valued function defined on X. Then F is LU-convex at $\mathbf{x}^*$ if and only if $\underline{f}$ and $\overline{f}$ are convex at $\mathbf{x}^*$.

The optimization problem is convex if all the functions and the feasible region are convex. A convex optimization problem is an important concept in the continuation. Chankong and Haimes [5] showed that if $\widehat{\mathbf{x}} \in X$ is a solution of problems (P_1) and (P_2), then there exists $0 \leq \lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_1 + \lambda_2 = 1$, such that $\widehat{\mathbf{x}}$ is also a solution of following problem

$$\begin{aligned}&\min \quad \lambda_1 \underline{f}(\mathbf{x}) + \lambda_2 \overline{f}(\mathbf{x}) \\ &\text{subject to:} \\ &g_i(\mathbf{x}) \leq 0, \;\; i = 1, ..., m\end{aligned}$$

provided that the optimization problem is convex.

4 Fritz-John Necessary Condition

Next, we showed Fritz-John necessary condition for a LU-solution of the problems with interval-valued objective function using E-differentiability.

Theorem 4.1. *Suppose the interval-valued function* $F : \mathbb{R}^n \to \mathbb{I}(\mathbb{R})$ *is continuous E-differentiable at* $\widehat{\boldsymbol{x}}$ *and LU-convex, and that the real-valued function* $g : \mathbb{R}^n \to \mathbb{R}^m$ *is continuous differentiable at* $\widehat{\boldsymbol{x}}$ *and convex. If* $\widehat{\boldsymbol{x}} \in X$ *is a LU-solution, then there exists (Lagrange) multipliers* $0 \leq (\boldsymbol{\lambda}, \boldsymbol{\mu}) \in \mathbb{R}^{2+s}$ *with* $(\boldsymbol{\lambda}, \boldsymbol{\mu}) \neq (\mathbf{0}, \mathbf{0})$ *such that*

$$\lambda_1 \nabla \underline{f}(\widehat{\boldsymbol{x}}) + \lambda_2 \nabla \overline{f}(\widehat{\boldsymbol{x}}) + \sum_{j=1}^{s} \mu_j \nabla g_j(\widehat{\boldsymbol{x}}) = 0.$$

DOI: 10.5540/03.2018.006.01.0369

Proof. If $\widehat{\mathbf{x}}$ is a LU-solution, then, by the Theorem 3.1, $\widehat{\mathbf{x}}$ is a solution of the problems (P_1) and (P_2). Supposedly, F and g are LU-convex and convex, respectively. This way, the optimization problem is convex, by Chankong and Haimes [5], we have that there exists $0 \leq \alpha_1, \alpha_2 \in \mathbb{R}$ and $\alpha_1 + \alpha_2 = 1$, such that $\widehat{\mathbf{x}}$ is also a solution of following problem

$$\begin{array}{ll} \min & \alpha_1 \underline{f}(\mathbf{x}) + \alpha_2 \overline{f}(\mathbf{x}) \\ \text{subject to:} & \\ g_i(\mathbf{x}) \leq 0, \;\; i = 1, ..., m. & \end{array}$$

Now, by the Fritz-John necessary condition for this problem, there exists $0 \leq (\widetilde{\boldsymbol{\lambda}}, \boldsymbol{\mu}) \in \mathbb{R}^{1+s}$ with $(\widetilde{\boldsymbol{\lambda}}, \boldsymbol{\mu}) \neq (\mathbf{0}, \mathbf{0})$ such that

$$\widetilde{\lambda} \nabla (\alpha_1 \underline{f} + \alpha_2 \overline{f})(\widehat{\mathbf{x}}) + \sum_{j=1}^{s} \mu_j \nabla g_j(\widehat{\mathbf{x}}) = 0.$$

The proof is completed considering $\boldsymbol{\lambda} = (\widetilde{\lambda}\alpha_1, \widetilde{\lambda}\alpha_2)$. □

5 Conclusion

We considered the LU order relation on the interval space. We used the E-differentiable for interval-valued function to obtain Fritz-John necessary condition for optimization problem with interval-valued objective function.

Acknowledgments

The research in this paper has been partially supported by both FAPESP-Brazil under the grants $(2012/00189-3$ and CEPID $2013/07375-0$ of Industrial Mathematics) and CNPq under the grants $(309335/2012-4$ and $479109/2013-3)$ and Capes Foundation under the grant (BEX $11153/13-0$).

References

[1] J. Aubin and A. Cellina, *Differential inclusions*, Springer-Verlag, New York, 1984.

[2] B. Bede and S. G. Gal, Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations, *Fuzzy Sets and Systems*, **151** (2005), 581–599.

[3] Y. Chalco-Cano, W. A. Lodwick and A. Rufin-Lizana, Optimality conditions of type KKT for optimization problem with interval-valued function via generalized derivative, *Fuzzy Optim Decis Making*, **12** (2013), 305–322.

[4] S. Chanas and D. Kuchta, Multiobjective programming in optimization of interval objective functions - a generalized approach, *European Journal of Operation Research*, **94** (1996), 594–598.

[5] V. Chankong and Y. Y. Haimes, *Multiobjective Decision Making: Theory and Methodology*, 1^{nd} edition, Elsevier Science Publishing, New York, 1983.

[6] M. Hukuhara, Intégration des applications mesurables dont la valeur est un compact convex, *Funkcial Ekvac*, **10** (1967), 205–229.

[7] H. Ishibuchi and H. Tanaka, Multiobjective programming in optimization of the interval objective function, *European Journal of Operational Research*, **48** (1990), 219–225.

[8] A. Rufián-Lizana, Y. Chalco-Cano, R. Osuna-Gómez and G. Ruiz-Garzón, On invex fuzzy mappings and fuzzy variational-like inequalities, *Fuzzy Sets and Systems*, **200** (2012), 84–98.

[9] L. Stefanini and B. Bede, Generalized Hukuhara differentiability of interval-valued functions and interval differential equations, *Nonlinear Analysis*, **71** (2009), 1311–1328.

[10] L. Stefanini and B. Bede, Some notes on generalized Hukuhara differentiability of interval-valued functions and interval differential equations, *Elsevier Science*, (2012), 1–12.

[11] H. C. Wu, Evaluate fuzzy optimization problems based on biobjective programming problems, *Computer and Mathematics with Applications*, **47** (2004), 893–902.

[12] H. C. Wu, The Karush Kuhn Tucker optimality conditions in an optimization problem with interval-valued objective function, *European Journal of Operation Research*, **176** (2007), 47–59.

[13] H. C. Wu, The Karush Kuhn Tucker optimality conditions in multiobjective programming problems with interval-valued objectivefunctions, *European Journal of Operation Research*, **196** (2009), 49–60.

[14] D.H. Martin, The essence of invexity, *Journal of Optimization Theory and Applications*, **47** (1985), 65–76.

DOI: 10.5540/03.2018.006.01.0369