Proceeding Series of the Brazilian Society of Computational and Applied Mathematics

Optimization Based Output Feedback Control Design in Descriptor Systems

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This article proposes an algorithm to minimize the Frobenius norm of output feedback matrix of a regular, linear time invariant, continuous time descriptor system. The resulting gain matrix ensures that the closed loop system is impulse-free and the associated non singular matrix is well-conditioned. By characterizing a subset of the set of non singular matrices through a linear matrix inequality, the related optimization is formulated as a semi-definite program.

Keywords. Output Feedback, Descritor Systems, Stabilization.

1 Introduction

This paper deals with the problem of stabilization by static output feedback for linear descriptor systems. Remind that an $n-$dimensional descriptor system consists of a mixture of $n-q$ algebraic equations and q first order diferential equations. Descriptor systems arise naturally in the modelling of several dynamical systems commonly used in engineering applications, such as biological system, power systems and other interconnected systems [12]. Besides guaranteeing the closed-loop asymptotical stability, two other properties are desired in practice: closed-loop regularity and absence of impulsive behavior. The problem of computing a suitable static output feedback, from which these closed-loop propierties are verified, is called simply stabilization problem. These three desired properties can be

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described in terms of the closed-loop eigenstructure: (i) the asymptotic stability is equivalent to have all the finite poles in the left half complex plane; (ii) the absence of impulsive modes is equivalent to have q finite closed-loop; and (iii) the regularity is guaranteed if the system is impulse-free. Thus, the necessary and sufficient conditions for the existence of a S-stabilizing output feedback are obtained as a set of coupled (generalized) Sylvester equations in [9], [7] . In case of normal systems, it is shown that output stabilizable (C, A, B)-invariant subspaces are obtained through a pair of coupled Sylvester equations for systems verifying Kimura's condition, thus two algorithm are proposed to solve the coupled Sylvester equations in [6], [14].

Since the impulsive response is undesirable in practical applications. One of the effective ways to eliminate impulses from a linear time invariant (LTI) descriptor system is by designing a suitable output feedback matrix that reduces the nilpotent index to one or strangeness index to zero (see [12], [9], [10] and the references therein). In the process, one has to select a output feedback gain matrix such that an appropriate sub-matrix, in the closed loop, would be non singular. It has been shown in [12], [9] that the inverse of resulting non singular matrix plays important role while designing a output feedback control for assigning the closed loop finite poles at appropriate locations in the complex plane. Hence, the non singular matrix should be well-conditioned to avoid numerical errors in the finite pole placement. The choice of feedback output matrix that eliminates impulses from the response of a descriptor system is not unique [12], [10], [9]. On the other hand, a limited magnitude of control signal can be provided to the actuators and the cost of the actuator grows quickly with increasing the amplitude of control signal. Since the control signal amplitude is directly proportional to the norm of the output feedback matrix, the minimum norm output feedback matrix satisfying the desired objectives would be preferable [13].

In this article we propose a convex algorithm to minimize the norm of output feedback matrix for an LTI regular descriptor system, ensuring that i) the response of the closed loop system is impulse free and ii) the associated non singular matrix is well-conditioned. To obtain a well-conditioned non singular matrix, we formulate a linear matrix inequality (LMI) optimization that minimizes the largest singular value and maximizes the smallest singular value of non singular matrix, simultaneously. Although there are several existing algorithms for impulse elimination, to the best of our knowledge, no results in the literature available that combines the above objectives and formulated it as a convex optimization in [8]. Rest of the paper is organized as follows. In Section II we formulate the problem following to some preliminaries on descriptor systems. In Section III we formulate the optimization to minimize the norm of output matrix, while at the same time improve the condition number of associated non singular matrix. Finally, we conclude in Section IV.

Notations: The set of all $m \times n$ constant real matrices is denoted as $R^{m\times n}$. I_n stands for an identity matrix of size $n \times n$. The Frobenius norm of $A \in R^{m\times n}$ is denoted as $||A||_K$, and defined as $||A||_K := \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2}$. $A > 0$ ($A \geq 0$) denotes that A is a symmetric positive definite (symmetric positive semi-definite) matrix.

2 Problem formulation

Let us consider an LTI continuous time descriptor system represented by the following equation:

$$E\dot{x} = Ax + Bu \tag{1}$$
$$y = Cx \tag{2}$$
$$x(t_0) = x_0,$$

where, if algebraic constraints are present, $E \in R^{n\times n}$ is singular, $A \in R^{n\times n}$, and $B \in R^{n\times m}$, $x : R \longrightarrow R^n$ is the state vector with initial condition $x(t_0) = x_0$ and $u : R \longrightarrow R^m$ is an input to the system. We assume that system (1) is regular, i.e., there exists a complex number $s \in C$ such that $det(sE - A) \neq 0$ (where $det()$ stands for determinant of a matrix). We refer to the roots of polynomial $det(sE - A)$ as finite poles of system (1). Furthermore, we assume that system (1) is impulse controllable and the involved matrices have appropriate dimensions. It is also assumed that B is full column-rank, C is full row-rank and that (C, A, B) is stabilizable and detectable.

The studied problem is to find a static output feedback control law

$$u(t) = Ky(t), \tag{3}$$

such that $\sigma(E, A + BKC) \in C^-$ or, equivalently, the closed-loop system is asymptotically stable. Considered the system

$$E_1\dot{x}(t) = A_1x(t) + B_1u(t) \tag{4}$$
$$0 = A_2x(t) + B_2u(t). \tag{5}$$

Let $V \in R^{v\times v}$ be such that $span(V) = V$ and $T \in R^{n-v\times v}$ be a left annihilator of V , i.e, $KerTE = ImV$. The following theorem relates the existence of a stabilizing static output feedback control law (3), to the solution of two coupled Sylvester equations. The system (4) is impulse controllable, *i.e.* $rank[E\, AS_\infty\, B]$, where the coluns of the matrix S_∞ span the kernel of E. Then there exists a output feedback K such that $u = Ky$ where $K \in R^{m\times p}$, such that the closed loop system:

$$E\dot{x} = (A + BKC)x, \tag{6}$$

would be impulse free, equivalently,

$$E_1\dot{x}(t) = A_1x(t) + B_1K_1C_1u(t) \tag{7}$$
$$0 = A_2x(t) + B_2K_2C_2u(t), \tag{8}$$

is impulse-free and the matrix $H_K = \begin{bmatrix} E_1 \\ A_2 + B_2K_2C_2 \end{bmatrix} \in R^{n\times n}$ is non singular. Since E_1 is full row rank: $rank(E_1) = q$, there exists an orthogonal matrix $W \in R^{n\times n}$ such that $H_W := HW$ would be in the following form (obtained by QR decomposition):

$H_W = \begin{bmatrix} E_1 \\ A_2 \end{bmatrix} W = \begin{bmatrix} E_1 & 0 \\ A_{21} & A_{22} \end{bmatrix}$, where $E \in R^{q\times q}$ is non singular and $A_{22} \in R^{n-q\times n-q}$. Now, by defining the output feedback matrix K as follows: $K = \begin{bmatrix} 0 & K_2C_2 \end{bmatrix} W^T$, where $K_2 \in R^{n-q\times n-q}$, we obtain the following relation: $H_K W = \begin{bmatrix} E_1 \\ A_2 \end{bmatrix} W + \begin{bmatrix} 0 \\ B_2K_2C_2 \end{bmatrix} W = \begin{bmatrix} E_{11} & 0 \\ A_{21} & A_{22}+B_2K_2C_2 \end{bmatrix}$, where $A_{22}+B_2K_2C_2 \in R^{n-q\times n-q}$. Note that the matrix E_{11} is non singular, and hence to make the closed loop system impulse-free we have to ensure that $\det(A_{22}+B_2K_2C_2) \neq 0$. Furthermore, it is demonstrated in [12], [9] that the inverse of matrix $A_{22}+B_2K_2C_2$ is needed while computing output feedback matrix for assigning the closed loop finite poles at appropriate locations in the complex plane. Hence, to avoid numerical errors in the finite pole placement, special care must be taken while computing K_2C_2, so as to ensure that $A_{22}+B_2K_2C_2$ is well-conditioned. In this context, the problem of interest can be precisely posed as follows.

Problem 1: Find $inf||K||_K$ such that i) response of the closed loop system (6) is impulse-free that is, $\det(A_{22}+B_2K_2C_2) \neq 0$ and ii) the condition number of $A_{22}+B_2K_2C_2$ is small.

3 Controller Design

In this section we formulate a semi-definite program to compute a desired output feedback matrix K which will ensure that the closed loop system (6) is impulse free and such that the matrix $A_{22}+B_2K_2C_2$ is well-conditioned. For this purpose, let us define a set N as follows:

$$N = \{K_2 \in R^{m\times n-q}/det(A_{22}+B_2K_2C_2) \neq 0\}. \quad (9)$$

Problem 2: Find $min_{K_2\in N}||K||_K$ such that the matrix $A_{22}+B_2K_2C_2$ is well-conditioned. Note that Problem 2 is a non-convex optimization since the set N is non-convex. To convexity the problem we establish a sufficient condition on K_2 in the following result based in [8] .

Theorem 3.1. *Let us denote $A_{K2} = A_{22}+B_2K_2C_2$. If K_2 satisfies following condition:*

$$\begin{bmatrix} \frac{1}{2}(A_{K2}+A_{K2}^T) & \beta I_{n-q} \\ \beta I_{n-q} & I_{n-q} \end{bmatrix} \succ 0, \quad (10)$$

for some positive β then $A_{K2} \in R^{(n-q)\times(n-q)}$ is non singular.

Proof: According to the Schur complement relation, (9) can be written as follows:
$\frac{1}{2}(A_{K2}+A_{K2}^T) - \beta^2 I_{n-q} \succ 0 \Longrightarrow \lambda_{min}(\frac{1}{2}(A_{K2}+A_{K2}^T)) > \beta^2$
$\Longrightarrow sn(A_{K2}) \geq \lambda_{min}(\frac{1}{2}(A_{K2}+A_{K2}^T)) > \beta^2$ where $sn(A_{K2})$ is the smallest singular value of A_{K2} and $\lambda_{min}(\frac{1}{2}(A_{K2}+A_{K2}^T))$ is the minimum eigenvalue of $\frac{1}{2}(A_{K2}+A_{K2}^T)$. Since the smallest singular value of A_{K2} is greater than zero, the matrix A_{K2} is non singular. This completes the proof.

Note that Theorem (3.1) defines a sufficient condition on K_2 such that the matrix A_{K2} is non singular. Additionally, inequality (9) is an LMI in β and K_2. Let us define a set M as follows:

$$M \quad = \quad \{K_2 \in R^{m\times n-q}/LMI \ (10) \ holds \ for \ \beta > 0)\}. \tag{11}$$

Then, M is a convex set; furthermore, $M \subseteq N$. Hence, instead of using a non convex set in Problem 2, we perform optimization over a convex set M. Next, we discuss on formulating a semi-definite program to minimize $||K||_K$.

Let us denote k_i^T for $i = 1, 2, \ldots, m$ as the i^{th} row of K_2. Then, $||K||_K$ can be minimized resulted in a positive scalar λ which satisfies an LMI:

$$\begin{bmatrix} I & k \\ k^T & \lambda \end{bmatrix} \succ 0,$$

where $k = [k_1^T, k_2^T, \ldots, k_m^T]^T$. This can be seen as follows: applying the Schur complement formula, we have

$$k^T k < \lambda \Longrightarrow ||K_2||_K < \lambda.$$

Since W is orthogonal, we can write

$$||K||_K = || \begin{bmatrix} 0 & K_2 \end{bmatrix} W^T ||_K = ||K_2||_K < \lambda.$$

Hence, to compute minimum norm K that eliminate impulses from the response of the closed loop system (4), we formulate the following convex optimization:

Problem 3: Find $min_{k,\beta,\lambda}\lambda$ subject to i) $\begin{bmatrix} \frac{1}{2}(A_{K2} + A_{K2}^T) & \beta I_{n-q} \\ \beta I_{n-q} & I_{n-q} \end{bmatrix} \succ 0$; ii) $\begin{bmatrix} I & k \\ k^T & \lambda \end{bmatrix} \succ 0$ for $\beta > 0$.

The solution of Problem 3 will produce a non singular matrix $A_{22} + B_2K_2C_2$, and hence the closed loop system (4) would be impulse-free. However, with this setting, since we are allowing β to take any arbitrary small positive number, the smallest singular value of A_{K2} (sn(A_{K2})) will also be very small. Then, the condition number of A_{K2},

$$k(A_{K2}) = \frac{s_1(A_{K2})}{sn(A_{K2})}, \tag{12}$$

where $s1(A_{K2})$ is the largest singular value of A_{K2}, would be very large, and hence the resulting non singular matrix A_{K2} might be ill-conditioned. We formulate an optimization where $sn(A_{K2})$ is maximized and $s1(A_{K2})$ is minimized simultaneously. This strategy will help us in improving the condition number of A_{K2}. We have already shown in Theorem 1 that $sn(A_{K2}) \geq \beta^2$, and hence maximization of β will maximize A_{K2}. Now, we will show that minimization of λ essentially reduces the maximum singular value of A_{K2} ($s1(A_{K2})$). Note that $s1(A_{K2})$ can be written as follows:

$$s_1(A_{K2}) = s_1(A_{22} + B_2K_2C_2) \tag{13}$$
$$\leq s_1(A_{22}) + (s_1(B_2)s_1(K)s_1(C_2)) \tag{14}$$
$$\leq s_1(A_{22}) + s_1(B_2)\lambda s_1(C_2), \tag{15}$$

and hence $s1(A_{K2})$ can be reduced by minimizing λ. The above inequalities follows from [11, Theorem 3.3.16]. Hence, a relaxed optimization associated with Problem 2 can be formulated as follows:

Problem 4: Find $max_{k,\beta,\lambda}\beta - \lambda$ subject to i) $\begin{bmatrix} \frac{1}{2}(A_{K2} + A_{K2}^T) & \beta I_{n-q} \\ \beta I_{n-q} & I_{n-q} \end{bmatrix} \succ 0$; ii) $\begin{bmatrix} I_{mn} & k \\ k^T & \lambda \end{bmatrix} \succ 0$ for $\beta > 0$.

Since the constraints (i) and (ii) in Problem 4 are $LMIs$ and the objective function is linear, it is an LMI optimization problem, and hence can be solved by existing standard LMI solvers see [8]. Once K_2 is computed by solving Problem 4, the desired matrix K, which makes the closed loop system (4) impulse free, can be obtained from the relation (7). The optimization Problem 4 is always feasible since we have assumed that the system (1) is impulse controllable.

Remark 3.1. *Recall that the set N , defined in (9), is a non convex set. Hence, to convexity the related optimization we compute a set M, in (9), which is a convex subset of N , and perform optimization over M. As a result the optimal (sub-optimal) solution of Problem 4 is an upper bound for the minimum value of $||K||_K$. In the following section we consider a numerical example to verify the effectiveness of the proposed approach.*

4 CONCLUSION

In this work we develop a novel algorithm to compute a output feedback matrix for an LTI regular descriptor system to eliminate impulses from the response. We represent a subset of the set of non singular matrices by an LMI, which enables to formulate the associated problem as a semi-definite program. We demonstrated through numerical example that the objective of only minimizing the norm of gain matrix might produce numerical errors while assigning the closed loop finite poles. This difficulty is then overcome by defining a cost function which simultaneously minimizes the norm of gain matrix and improve the condition number of the associated non singular matrix. The proposed method can be extended in the direction of developing a convex algorithm to minimize the norm of output feedback matrix for assigning the finite poles at i) some fixed locations ii) within a stability region in the complex plane. These objectives are under current investigation.

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DOI: 10.5540/03.2018.006.01.0410