Proceeding Series of the Brazilian Society of Applied and Computational Mathematics, v. 6, n. 1, 2018.

Trabalho apresentado no XXXVII CNMAC, S.J. dos Campos - SP, 2017.

Proceeding Series of the Brazilian Society of Computational and Applied Mathematics

Numerical solutions of the Orr-Sommerfeld equation for a thin liquid film on an inclined plane.

Bruno Pelisson Chimetta[1]
University of Campinas, Faculty of Mechanical Engineering, Rua Mendeleyev, 200, CEP 13083-860, Campinas, SP
Erick de Moraes Franklin[2]
University of Campinas, Faculty of Mechanical Engineering, Rua Mendeleyev, 200, CEP 13083-860, Campinas, SP

Abstract: A numerical solution of the initial behavior of free surface liquid flows on inclined planes is presented. These instabilities, under some conditions, may evolve to surface-wave, that often appear on thin liquid films. Liquid films help us to remove the heat from solid surfaces, and also reduce the friction between high viscosity fluids and pipe walls, among other applications; therefore, understanding and predicting this behavior is useful in industry. The surface-waves instability phenomena are governed by the Orr-Sommerfeld equation and their boundary conditions. In this work we present a study through a numerical approach of the Orr-Sommerfeld equation. The numerical solution was based on a Galerkin method using Chebyshev polynomials for the discratization, which made it possible to express the Orr-Sommerfeld equation and their boundary conditions as a generalized eigenvalue problem. The method is compared with previous works for validation. The solution gives the critical conditions in which the liquid film turns unstable, and describes possible features that produce these instabilities. All codes, data and plots were produced in the MATLAB environment.

Keywords: Surface waves, Orr-Sommerfeld equation, Galerkin Method, Chebyshev Polynomials.

1 Introduction

This work is devoted to the initial behavior of liquid films flowing over an inclined plane. Instabilities often appear on the free surface of liquid flows. Such knowledge is useful in industrial applications, once liquid films help us to remove, or increase, heat and mass transfers from solid surfaces, such as in reactors cooling processes. Friction reducing effects are also important. The surface-wave instabilities are governed by the Orr-Sommerfeld equation [7, 9] and their boundary conditions. As boundary conditions of the problem, it was considered the no-slip condition at the wall, kinematic condition of the interface and dynamic condition of the interface. To obtain the Orr-Sommerfeld equation,

[1]brunopchimetta@gmail.com
[2]franklin@fem.unicamp.br

the authors introduced perturbations in the form of a stream function in the Navier-Stokes equation, as done by [11], then used normal modes for these functions. Similar approach for the development of the Orr-Sommerfeld equation can be found in [8]. After a linearization of the perturbations, we obtain the Orr-Sommerfeld equation. For the final form of the boundary conditions the same procedure is applied. The numerical solution was based on a Galerkin method [5] using Chebyshev polynomials for the discretization, which made it possible to express the Orr-Sommerfeld equation and their boundary conditions as a generalized eigenvalue problem. All those choices were made because of the general approach provided by the Galerkin method, which makes the implementation of the boundary conditions of free surface easier, and the high accuracy of the Chebyshev polynomials. A code was implemented in Matlab to solve the eigenvalue problem using a QZ algorithm. The main goal of this work is to reach the critical conditions in which the liquid film flow turns unstable, and to express the results in terms of the dimensionless groups Reynolds and Froude. The results are compared with previously published data.

2 Formulation of the problem and the Orr-Sommerfeld equation

The development of the momentum equations presented next are based on [4]. It was considered a film of incompressible Newtonian fluid of viscosity μ, density ρ, and thickness h, falling down on a inclined plane with an angle θ with respect to the horizontal. The interface between the liquid and the gas has a surface tension γ and no surfactants are present. The pressure applied by the gas on the interface is P_0. For the base state the interface $H(x, y, t)$ and the position of the interface $\eta(x, t)$ are both planar. Based on the Navier-Stokes equations and considering the boundary conditions of permanent flow, non-slip at solid surface, no shear, constant pressure at the liquid-gas interface and without presence of a gradient temperature, a solution corresponding to a steady parallel flow, with a planar interface and parabolic velocity profile, was found. The velocity of the interface U_0, present in the velocity profile, is given by,

$$U_0 = \frac{\rho g h^2 sin(\theta)}{2\mu} \tag{1}$$

The contribution of inertia related to viscosity, gravity, and surface tension are measured, respectively, by the Reynolds, Froude, and Weber numbers. The solutions are given in terms of the parameters h, ρ and U_0. The dimensionless groups are defined as:

$$Re = \frac{\rho U_0 h}{\mu}, \quad Fr = \frac{U_0^2}{ghcos(\theta)} = \frac{Re\ \tan(\theta)}{2}, \quad We = \frac{\rho U_0^2 h}{\gamma} \tag{2}$$

The Froude number is defined using the gravity component $gcos(\theta)$ normal to the flow and γ is the surface tension. When the interface is perturbed ($\eta(x, t) \neq 0$), the velocity profile no longer has an exact parabolic behaviour. This, combined with inertia, leads to surface waves instabilities. In order to find the critical conditions for these waves and reach the Orr-Sommerfeld equation, and the boundary conditions of the problem, it was considered only two dimensional perturbations. This choice is based on the Squire's theorem which sets that [3],

Theorem 2.1. *To obtain the minimum critical Reynolds number it is sufficient to consider only two-dimensional disturbances.*

This theorem remains valid for a flow with interface [6]. It was considered only the two-dimensional case for the Navier-Stokes and the continuity equations, and first order perturbations as well. For the perturbations, stream functions were used, and a cross differentiation between the equations of motion, for x and y components, was applied. The normal mode for the stream functions is given by,

$$\Psi(x,y,t) = \hat{\Psi}(y)e^{i\alpha(x-ct)} \tag{3}$$

where $c = \frac{\omega}{k} \in \mathbb{C}$, $\alpha = kh \in \mathbb{R}$, k is the wave number; therefore, α is the characteristic wave number of the problem, and ω is the frequency. Using $\Psi(x,y,t)$ and making the contraction of notation for $D = \frac{\partial}{\partial y}$, we obtain the Orr-Sommerfeld Equation in dimensionless form, given by:

$$(D^2-\alpha^2)^2\hat{\Psi}(y) = i\alpha Re[(\overline{U}-c)(D^2-\alpha^2) - D^2\overline{U}]\hat{\Psi}(y) \tag{4}$$

For the boundary conditions of the problem, which are, wall condition, kinematic condition of the interface and dynamic condition of the interface, the same procedure used for Orr-Sommerfeld equation was applied. The wall condition is given by:

$$D\hat{\Psi}(-1) = 0 \tag{5}$$

$$\hat{\Psi}(-1) = 0 \tag{6}$$

The kinematic condition is a function of $\eta(x,t)$ and $\Psi(x,y,t)$; therefore, its necessary to consider the normal mode for the interface position $\eta(x,t)$, as done in Eq. (3). After applying a linearization around $y = 0$, was found,

$$\hat{\Psi}(0) - (c-1)\hat{\eta} = 0 \tag{7}$$

where $\hat{\eta}$ is the amplitude of deformation of the interface. The dynamic condition at the interface has two parts. The first condition is the continuity of the tangential stress, associated with the viscous stress of the fluid. The second one is the continuity of the normal stress associated with the surface tension. Reducing the effect of air to a purely normal stress, applying the normal modes and linearizing around $y = 0$, we obtain:

$$D^2\hat{\Psi}(0) + \alpha^2\hat{\Psi}(0) + \hat{\eta}D^2\overline{U}(0) = 0 \tag{8}$$

$$-D^3\hat{\Psi}(0) + [3\alpha^2 - i\alpha Re(c-1)]D\hat{\Psi}(0) + i\alpha Re\left[\frac{1}{Fr} + \frac{\alpha^2}{We}\right]\hat{\eta} = 0 \tag{9}$$

3 Numerical approach

For the numerical solution the authors implemented a Galerkin method using Chebyshev polynomials of the first kind, known as T_n, for the discretization. In order to implement the method it is necessary transfer the problem domain to the interval $[-1; 1]$, because of the orthogonal properties of the Chebyshev polynomials in this interval. The transformation applied was, $z = 2y + 1$ for $y \in [-1; 0]$. Rearranging the boundary conditions in order to eliminate $\hat{\eta}$ and tanking $\hat{\Psi}(z)$ as an approximation given by,

$$\hat{\Psi}(z) = \sum_{k=0}^{N} a_k T_k(z); \quad k \in \{\mathbb{Z} \ | \ k \geq 0\} \tag{10}$$

it is possible to write the Orr-Sommerfeld equation in terms of the inner products required in the Galerkin method. After this process, we can rewrite the problem as an eigenvalue problem in the form,

$$[\mathbf{A}]_{NxN}\vec{a} = c[\mathbf{B}]_{NxN}\vec{a} \tag{11}$$

where N is the number of Chebyshev polynomials to be used and the matrices $\mathbf{A}$ and $\mathbf{B}$ can be written respectively as $\mathbf{A} = A_r + iA_i$ and $\mathbf{B} = B_r + iB_i$. To apply the boundary conditions the authors used the same approximation given by Eq. (10) for $z = -1$ for wall conditions and $z = 1$ for the interface conditions, and then replaced the last lines in the Eq. (11) with the transformed boundary conditions. In the Galerkin method it is possible to use the boundary conditions as line vectors in the final matrices, making the implementation easier. The choice of using Chebyshev polynomials was made because of their high accuracy, and their orthogonal properties, which makes the implementation easier. A code was written in the MATLAB environment to solve the Eq. (11). We used the function 'eig' which uses a Cholesky factorization or a generalized Schur decomposition (QZ algorithm) based on the properties of $\mathbf{A}$ and $\mathbf{B}$. If $\mathbf{A}$ and $\mathbf{B}$ are symmetric, the standard choice will be the Cholesky factorization, otherwise the software will implement the QZ algorithm. All codes implemented in this work were made using a number of Chebyshev polynomials equal to 80, except those present in table 1.

4 Discussion and results

As was mentioned in section 2, $c \in \mathbb{C}$; therefore, $c = c_r + ic_i$ were c_r and $\sigma = \alpha c_i$ are the phase velocity and the growth rate of the wave respectively. If $c_i < 0$ the system is stable, and if $c_i > 0$ the system is unstable, following the formulation adopted for the normal modes in this work. This approach treats the wave as a small perturbation, which will provoke an instability at the interface. Therefore, our main focus is to find the critical conditions for the growth rate σ, and the parameters that provoke these conditions. For the validation of the numerical method the authors used as reference a physical problem considered by [1]. For the implementation the following values were used $\theta = \pi/3$, $\alpha = 0.01$, $We = 0.0001$, $Re = 1$, the results are shown in table 1. All numerical results were obtained using 16 Chebyshev polynomials, and it is possible to see good agreement between both works. These results were obtained after a convergence of the physical eigenvalue, shown in table 1.

For the growth rate $\sigma(\alpha)$ presented in Fig. 1, the following properties were used $\mu = 0.001 Ns/m^2$, $\rho = 998.2071 Kg/m^3$, $g = 10 m/s^2$, $T = 20^o C$, 0.1 mm thickness and $\gamma = 0.07275 N/m$ as referenced in [10]. This figure shows the behaviour of the growth rate, as a function of α, for different values of the Froude number, with $Fr_c = 0.6250$ and $\theta_c = 26,40^o$, for the range $22,5^o < \theta < 31^o$.

Figure 2 presents the behaviour of the growth rate as a function of the Reynolds number, with $Re_c = 1,9262$ and the corresponding growth rate $\sigma = -1,5914.10^{-9}$, for the range $1 < Re < 2$. Figure 3 shows the stability diagram which separates the stable and unstable regions. The values shown close to each line represents the growth rate value σ of each line. The positive values represent the unstable region and the negative values the stable region, and they are limited by the neutral curve that represents $\sigma = 0$. The onset of the unstable band appears at $Re_c = 1,9262$ and $\sigma = -1,5914.10^{-9}$, the same values from Fig. 2. For both figures, it was considered $\alpha = 0.01$, $\theta = \pi/3$, $We = 0.0001$.

Table 1: Value of the physical eigenvalue based on the number of Chebyshev polynomials used. The results were compared with previous work for the validation of the method [1].

N	4	5	6	8	12	16
c_r	1,999756	1,999820	1,999818	1,999818	1,999818	1,999818
c_i	-0.051849	-0.301656	-0,185036	-0,185036	-0,185036	-0,185036

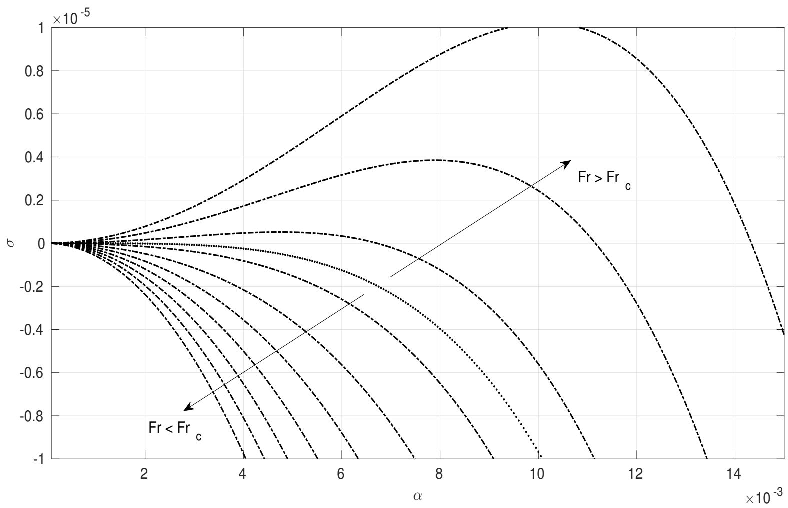

Figure 1: Behavior of the growth rate $\sigma(\alpha)$ for $Fr < Fr_c$ and $Fr > Fr_c$. The dotted line represents the growth rate for the critical Froude number.

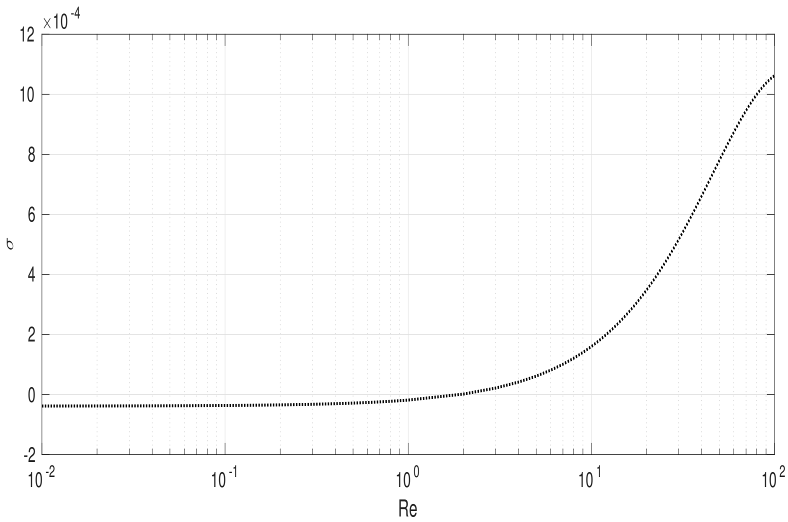

Figure 2: Behavior of the growth rate $\sigma(Re)$ for the numerical solution.

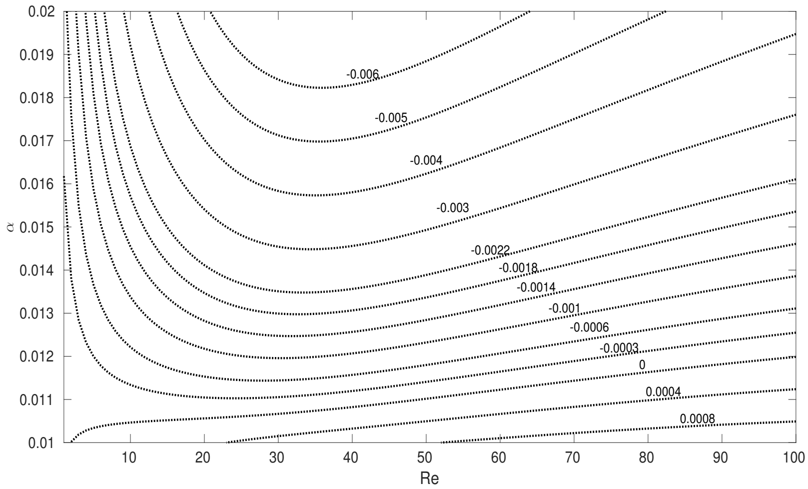

Figure 3: Stability diagram as a function of the Reynolds number for the interval $1 < Re < 100$.

5 Conclusion

With the Galerkin method it was possible to find, as shown in Fig. 2, the critical Reynolds number, which determines a critical condition when the inertial effects dominates the flow and the liquid film becomes unstable. This critical point has major importance once, after reaching the critical conditions, the flow patterns can change to other patterns less, or more, desirable than the previous one. As was presented in the stability diagram of the Fig. 3 for the Reynolds number, if the inertial effects increase, the unstable band

DOI: 10.5540/03.2018.006.01.0406

tends to increase. It was also possible to find other curves for the growth rate besides the neutral curve ($\sigma = 0$) in the instability diagram. This method shows itself as a good alternative to implement complex boundary conditions in an easier way in order to solve stability problems.

Acknowledgements

The authors are grateful to FAPESP (grant no. 2012/19562-6 and 2016/13474-9), to CNPq (grant no. 471391/2013-1), to FAEPEX/UNICAMP (conv. 519.292, projects AP0008/2013 and 0201/14), and to CAPES for the provided financial support.

References

[1] F. Charru, *Hydrodynamic Instabilities.* Cambridge University Press. 2011. DOI: 10.1017/CBO9780511975172.

[2] B. P. Chimetta and E. M. Franklin, Asymptotic solution of the Orr-Sommerfeld equation for surface waves on an inclined plane. *In 23rd ABCM International Congress of Mechanical Engineering.* ABCM, Rio de Janeiro, Brazil, 2015.

[3] P. G. Drazin and W. H. Reid, *Hydrodynamic stability.* Cambridge University Press, 2004. DOI: 10.1017/CBO9780511616938.

[4] J. A. Fay, *Introduction to fluid Mechanics.* MIT Press. 1994. ISBN 10: 0-262-06165-1.

[5] C. Fletcher, Computational Galerkin Methods. *Computational Physics Series.* Springer-Verlag. 1984. DOI: 10.1007/978-3-642-85949-6.

[6] T. I. Hesla, F. R. Pranckh and L. Preziosi, *Squire's theorem for two stratified fluids.* Physics of Fluids (1958-1988), Vol. 29, pp. 2808-2811, 1986. DOI: 10.1063/1.865478.

[7] W. M. Orr, *The stability or instability of steady motions of a liquid, part II: A viscous liquid.* Proc. Roy. Irish Acad. Sect. A, Vol. 27, pp. 69-138. 1907.

[8] R. L. Panton, *Incompressible flow.* John Wiley & Sons, 2013. DOI: 10.1002/9781118713075.

[9] A. Sommerfeld, Ein beitrag zur hydrodynamischen erklaerung der turbulenten fluessigkeitsbewegungen. *In Proceedings of the 4th International Congress of Mathematicians III*, Rome, Italy. pp. 116-124, 1908.

[10] N. Vargaftik, B. Volkov and L. Voljak, *International tables of the surface tension of water.* Journal of Physical and Chemical Reference Data, Vol. 12, pp. 817-820, 1983. DOI: 10.1063/1.555688.

[11] C. S. Yih, *Stability of liquid flow down an inclined plane.* Physics of Fluids (1958-1988), Vol. 6, pp. 321-334, 1963. DOI: 10.1063/1.1706737.

